A "Convolution" Method for Option Pricing I : European and Barrier Options

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Abstract

We present a new methodology for option pricing, using an approximation of the kernel of the Black & Scholes P.D.E. by step functions, instead of Dirac measures like in the Cox, Ross & Rubinstein binomial tree. The algorithm one can derive from this methodology converges very rapidly, as well for the prices, as for the derivatives Δ, Γ, ϑ, Θ, ρ and works with the same efficiency for European and barrier options. The principle can be applied to American options using the Mac Millan or Barone-Adesi & Whaley "semi-analytic" approximations. This will be the topic of the second part of this work.

1. Introduction

The so-called "binomial tree" algorithm, developed by Cox, Ross and Rubinstein (see [1]) is widely used in the pricing of American options. It has however a major drawback: the slowness of its convergence and the consequent amount of time required to achieve the risk analysis of big portfolios. Further more, in the case of *barrier options* ("up & out" options, etc.) the binomial tree not only converges rather slowly on prices, but simply does not converge at all on the Δ, Γ, etc[1].

[1]Some computational tricks may give rather good results on the Δ.

This phenomenon finds its origin in an incorrect use of the "central-limit" theorem. Assume that the price of the underlying asset, denoted S_t, follows a Log-normal diffusion process under the risk-neutral probability P:

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dW \tag{1.1}$$

For the sake of simplicity, we shall consider $x_t = \log \frac{S_t}{S_0}$, so that:

$$dx_t = \mu\, dt + \sigma\, dW \qquad\qquad \mu = r - \frac{\sigma^2}{2}$$

Denote by $p_{t,\delta t}(\delta x)$ the distribution density of $x_{t+\delta t} - x_t$ knowing x_t and by $p_{t+\delta t,\delta t}(\delta x)$ that of $x_{t+2\delta t} - x_{t+\delta t}$ knowing $x_{t+\delta t}$. Then, for small δt, the distribution of $x_{t+2\delta t} - x_t$ knowing x_t is given by the *convolution* of these two densities:

$$p_{t,2\delta t} = p_{t,\delta t} * p_{t+\delta t,\delta t}$$

or else:

$$p_{t,2\delta t}(\delta x) = \int_{-\infty}^{\infty} p_{t,\delta t}(u)\, p_{t+\delta t,\delta t}(\delta x - u)\, du$$

In the models with non stochastic volatilities, the functions $p_{t,\delta t}$ do not depend, or depend slowly on t, and their standard deviation is proportional to $\sqrt{\delta t}$. The *central-limit theorem* then justifies the use of Log-normal laws of probability (provided the variance is finite). Moreover, if the functions $p_{t,\delta t}$ are continuous, or at least piecewise continuous, and with compact support, then the convergence occurs uniformly and in the weak sense. Let $f(x)$ be the pay-out of an option with maturity T, and assume that f is continuous, then:

$$\lim_{n\to\infty} p_{t_1,\delta t} * \cdots * p_{t_n,\delta t} * f(x) = g * f(x) \tag{1.2}$$

where $\delta t = \frac{1}{n}T$, $t_{i+1} - t_i = \delta t$ and g is a suitable Gaussian function[2]. The speed of convergence is of the order of $\frac{1}{\sqrt{n}}$. The derivatives can be obtained simply by differentiating f.

In the binomial tree, $p_{t,\delta t}$ is not a function, but the sum of two Dirac distributions. The law of large numbers however ensures the convergence of the algorithm,

[2] If $f(t,s)$ is the solution to the Black & Scholes P.D.E., then g is the kernel of this parabolic equation.

provided f is piecewise continuous (see [2], pp.165-178). Hence this works for the price and the Δ but not further (in fact, one can compute the Γ for "plain vanilla" options).

The general idea of our method to compute option prices and risk analysis relies on four observations:

1. In general, pay-off functions are piecewise constant, affine or exponential, namely, on each interval one has:

$$f(x) = \begin{cases} a\,e^x + b & \text{(Log-normal distribution)} \\ \\ a\,x + b & \text{(Normal distribution)} \end{cases}$$

2. The momenta of the Gaussian kernel g are easily computed.

3. If the $p_{t,\delta t}$ are piecewise C^1 (not necessarily continuous) then $p_{t_1,\delta t} * \cdots * p_{t_n,\delta t}$ is of class C^{n-2} and the convergence (1.2) towards g occurs in the C^∞ topology, which makes possible, in order to calculate the derivatives Δ and Γ, to derive the convolution product and g instead of f.

4. The convolution between a function of the form *"exponential + polynomial"* on an interval, and zero outside the interval, with the characteristic function of another interval, is piecewise of the same form *"exponential + polynomial"*, the new polynomial having just one degree more than the previous one.

Assume now that the functions $p_{t,\delta t}$ are *step functions*. A convolution of n of these and f is a function which is piecewise of the form *"exponential + polynomial of degree n"*. A lattice is then defined in the time-price space, and the price of the option on each mesh of the underlying price lattice is given by an interpolation based on a combination of exponential and polynomial functions. This only requires to keep in memory a list of coefficients for each interval. A coefficient at time t is computed out of the coefficients at time $t+\delta t$. A backward recurrence gives in the end a formal expression of the price on every interval. Two consecutive formulas coincide at a node up to order $n-2$ and one can differentiate formally these expressions to get the derivatives Δ and Γ.

The complexity of the algorithm is in n^3. Each step requires more computing time than a step of a binomial tree, but the precision after 5 to 10 steps is equivalent to that of a binomial tree with several hundreds of steps.

This new method may also be useful for building a graphic tool to analyse global portfolio positions and study catastrophe scenarios. It works on barrier options by applying the mirror lemma and Girsanov theorem. The number of coefficients then increases by 2 at each time step.

American options will be the topic of part II of this work. The McMillan [4] or Barone-Adesi & Whaley[3] semi-analytic approximations of Black & Scholes PDE are used and the value of the option is given by a combination polynomials and of exponential functions with different exponents.

In this article, we give an upper bound of the error with respect to the number of steps and to the volatility in the case of European options. A more complete study of the errors and of the computing times will be given in a further paper. The theoretical error is in fact easier to compute than for the binomial tree, and accelerating procedures could be set up, if really needed.

2. European options, convolution with step functions

In this section, we first present a simplified version of the convolution method. It is convenient for European options with constant interest rate and volatility. Then we give the general method for European options. These versions have essentially a didactic purpose, because of the Black-Scholes closed formula, which gives an exact result.

2.1. Characteristic functions of intervals

Consider a European option (call or put) with maturity T and strike K on an asset whose price $S_t = S_0 e^{x_t}$ follows a diffusion process as above (see (1.1)) with constant μ and σ. Let $\varphi_t(x)$ be the price of the option at time t knowing that $x_t = x$ and $r = \mu + \frac{\sigma^2}{2}$ the (constant) risk less interest rate. Then, if $t < t' < T$, one has:

$$\varphi_t = e^{-r(t'-t)}\, g_{t'-t} * \varphi_{t'}$$

where $g_{t'-t}$ is a Gaussian function with mean value $-\mu(t'-t)$ and standard deviation $\sigma\sqrt{t'-t}$, that is:

$$g_{t'-t}(x) = \frac{1}{\sigma\sqrt{2\pi(t'-t)}}\, e^{-\frac{(x+\mu(t'-t))^2}{2\sigma^2(t'-t)}} \tag{2.1}$$

When $t' - t = \delta t$ is small, one can approximate $g_{\delta t}$ by a characteristic function of an interval. Fix the integer n, define $\delta t = \frac{T}{n}$ and:

$$p_{\delta t} = \frac{1}{2\sigma\sqrt{3\delta t}} \mathbf{1}_{[-\mu\delta t - \sigma\sqrt{3\delta t}\,,\, -\mu\delta t + \sigma\sqrt{3\delta t}]}$$

where $\mathbf{1}_I(x) = 1$ if $x \in I$, and $\mathbf{1}_I(x) = 0$ otherwise. Build a lattice in (t, x) with:

$$\begin{cases} t_i = i\,\delta t \quad , & i = 0, \ldots, n \\ x_j^i = \log \dfrac{K}{S_0} - i\,\mu\,\delta t + j\,\sigma\sqrt{3\delta t}\,, & j = -J, \ldots, J \quad \text{at } t = t_i \end{cases}$$

Set:

$$\tilde{\varphi}_T(x) = \varphi_T(x) = f(x) \qquad \text{and} \qquad \tilde{\varphi}_{t_i} = e^{-r\delta t}\,\tilde{\varphi}_{t_{i+1}} * p_{\delta t}$$

Then:

$$\tilde{\varphi}_{t_i}(x) = \frac{e^{-r\delta t}}{2\sigma\sqrt{3\delta t}} \left(\tilde{\Phi}_{t_{i+1}}(x + \mu\,\delta t + \sigma\sqrt{3\delta t}) - \tilde{\Phi}_{t_{i+1}}(x + \mu\,\delta t - \sigma\sqrt{3\delta t})\right)$$

where $\tilde{\Phi}_{t_{i+1}}$ is a primitive of $\tilde{\varphi}_{t_{i+1}}$. It is clear that, if f is of the form *exponential + constant,* then on each mesh $[x_j, x_{j+1}]$ at time t_i, then the function $\tilde{\varphi}_{t_i}$ is of the form:

$$\tilde{\varphi}_{t_i}(x) = a_{ij}\,e^x + Q_{ij}(x)$$

where Q_{ij} is a polynomial of degree $n - i$.

The principle of the algorithm consists in keeping in memory, at each step and on each mesh, the number a_{ij} and the coefficients q_{ijm} of the polynomial Q_{ij}. Then an induction formula in i is settled, which does not involve any computation of non rational function, and only at the end, the price of the option is computed with respect to the initial data.

Remark: In the induction formula, it is possible, if necessary, to make the interest rate vary from one step to another. However, if the volatility varies, then the "lattice" shape at each step t_i would no longer survive, for intervals would not any more match with their neighbors. This obstacle can be solved by adjusting the t_i's so that the standard deviation between two consecutive time steps be constant.

2.1.1. Theorem:

Let $\varphi_0(x)$ *be the initial price of the option when* $S_0 = K\,e^x$ *and set:*

$$\tilde{\varphi}_n(x) = \tilde{\varphi}_{t_0}(x)$$

when n *steps are used in time. Then:*

$$\lim_{n\to\infty} \tilde{\varphi}_n = \varphi_0$$

and the convergence occurs in the compact-open C^∞ *topology.*

Proof. This is proved like the central limit theorem. Let $p_n = p_{T/n}$ and:

$$\tilde{g}_n = p_n^{*n} = \underbrace{p_n * \cdots * p_n}_{n \text{ times}}$$

One has:

$$\tilde{\varphi}_n = e^{-rT}\,\tilde{g}_n * f \qquad \text{and} \qquad \varphi_0 = e^{-rT}\,g_T * f$$

Let now $\hat{g}_n$, $\hat{p}_n$ and $\hat{g}_T$ be the respective Fourier transform of $\tilde{g}_n$, p_n and g_T :

$$\hat{g}_n(s) = \int_{-\infty}^{\infty} \tilde{g}_n(x)\,e^{-isx} dx \quad , \qquad \text{etc.}$$

The following identities are easily checked:

$$\hat{g}_n = \hat{p}_n^n$$

and

$$\hat{p}_n(s) = \hat{p}_1\left(\frac{s}{\sqrt{n}}\right)$$

As p_1 has a compact support, the Fourier transform $\hat{p}_1$ is defined on the whole complex plane $\mathbf{C}$ and a Taylor expansion of $log\,\hat{p}_1(s)$ in 0 shows that, for any $s \in \mathbf{C}$:

$$\lim_{n\to\infty} n \log \hat{p}_n(s) = \frac{1}{2}\,\sigma^2 T\,s^2$$

In the case of a put option, the pay-out function f is bounded and its support is bounded from above, hence, for any $\varepsilon > 0$, the function $f(x)\,e^{\varepsilon x}$ is integrable on $\mathbf{R}$. In the case of a call option, then $f(x)\,e^{-(1+\varepsilon)x}$ is integrable. Thus there always exists an α such that $e^{\alpha x} f \in L^1(\mathbf{R})$ and the Fourier transform $\hat{f}$ is defined on a half-plane $\{\operatorname{Im} s < -1\}$ for a call and $\{\operatorname{Im} s > 0\}$ for a put.

One now knows that:

$$\tilde{g}_n * f\,(-x) = \frac{1}{2\pi} \left(\widehat{\hat{g}_n(s+i\alpha)\,\hat{f}}(s+i\alpha) \right)$$

The same equality holds with g_T and $\hat{g}_T$. For the derivatives:

$$(\tilde{g}_n * f)^{(k)}\,(-x) = \frac{1}{2\pi} \left(\hat{g}_n(s+i\alpha)\,\widehat{\hat{f}(s+i\alpha)\,(is)^k} \right)$$

The error on the price $(k=0)$ and on the derivatives is:

$$\begin{aligned}
((\tilde{g}_n - g_T) * f)^{(k)}\,(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\hat{g}_n(s+i\alpha) - \hat{g}_T(s+i\alpha))\,\hat{f}(s+i\alpha)\,(is)^k ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{n \log \hat{p}_n(s)} - 1 \right) \hat{f}(s+i\alpha)\,\hat{g}_T(s+i\alpha)\,(is)^k ds
\end{aligned}$$

Lebesgue theorem proves that this last integral tends to 0 as n tends to ∞, for $\hat{f}(s+i\alpha)$ is bounded and $\hat{g}(s+i\alpha)\,s^k \in L^1(\mathbf{R})$ for any $\alpha \in \mathbf{R}$ and any $k \in \mathbf{N}$. □

Note that, thanks to the Taylor expansion of $log\,\hat{p}_1$, the speed of convergence depends on the number of momenta in common between the step function $p_{\delta t}$ and the Gaussian function $g_{\delta t}$. As 3 of them coincide (for parity reasons), the algorithm converges towards the exact price like $\frac{1}{n}$ as n tends to $+\infty$. The derivatives Δ and Γ have almost the same speed of convergence (respectively $\frac{\sqrt{\log n}}{n}$ and $\frac{\log n}{n}$). This fact will be confirmed by the theorem 3.1.

For fixed n, the error is proportional to σ for the price, and tends towards a constant for the Δ. It is not bounded for the Γ.

Practically, if $\sigma = 10\%$ and $n = 10$, then the relative errors on prices, Δ and Γ are all of the order of 10^{-5} at the money.

2.2. Step functions

In order to generalise this algorithm to the case of barrier or American options, one has to change of convolution functions $p_{\delta t}$ while keeping as far as possible the simplicity of the algorithm. We shall therefore consider step functions instead of simply bump functions. We may assume, without loss of generality, that $S_0 = K$. Let:

$$x_j = j\,\delta x \qquad j \in \mathbf{Z} \qquad \delta x > 0$$

be a regular lattice in x (so that x_0 corresponds to the strike) and denote by ν be the number of steps of $p_{\delta t}$. We shall assume $\nu \geq 3$ and that the location of these steps is fixed by the lattice $(x_j)_{j\in\mathbf{Z}}$:

$$p_{\delta t} = \sum_{h=h_{\min}}^{h_{\max}} \omega_h\,\eta_h \tag{2.2}$$

with

$$h_{\min} = -\left\lfloor \frac{\nu}{2} \right\rfloor \qquad h_{\max} = h_{\min} + \nu - 1 \qquad \eta_h = \mathbf{1}_{[x_h, x_{h+1}]}$$

The momenta of $p_{\delta t}$ are required to fit those of the Gaussian $g_{\delta t}$ (defined by (2.1)) up to the maximal order, namely $\nu - 1$. This means:

$$\int_{-\infty}^{\infty} p_{\delta t}(x)\,x^j\,dx = \int_{-\infty}^{\infty} g_{\delta t}(x)\,x^j\,dx \qquad \forall j\,, \quad 0 \leq j \leq \nu - 1$$

These ν linear equations define the ω_h's. Notice that the determinant of this system can be reduced to "Van der Monde" type based on the nodes of the lattice, and thus never vanishes. The origin of the lattice in x needs not anymore to depend on time, because the shift due to the drift μ can now be taken into account in the value of the ω_h's.

In order to apply the central limit theorem, it is necessary that $\omega_j \geq 0$ for all j. This will imply some inequalities between the drift μ, the volatility σ and the sizes δt and δx of the meshes in t and x. In particular, δx must be of the order of $\sigma\sqrt{\delta t}$.

Remark: If r is very large with respect to σ, or if δt is big, one could have to shift the range of h in order to keep the coefficients ω_h non negative. Nevertheless, in current numerical situations, such an eventuality does not occur.

2.3. Induction formula

Now, when x belongs to a mesh $[x_j, x_{j+1}]$, we set:

$$\tilde{\varphi}_{t_i}(x) = a_{ij}\, e^x + \sum_{m=0}^{n-i+1} b_{ijm}\,(x - x_j)^m$$

which we may write as:

$$\tilde{\varphi}_{t_i}(x) = \sum_{j=-J}^{J} a_{ij}\,\eta_j(x)\,e^{x-x_j} + \sum_{j=-J}^{J}\sum_{m=0}^{n-i+1} b_{ijm}\,\eta_j(x)\,(x - x_j)^m$$

One can check that:

$$\begin{aligned}
\left(e^{x-x_j}\,\eta_j\right) * \eta_h &= \left(e^{x-x_{j+h}} - 1\right)\eta_{j+h} + \left(e^{\delta x} - e^{x-x_{j+h+1}}\right)\eta_{j+h+1} \\
((x-x_j)^m\eta_j) * \eta_h &= \frac{(x - x_{j+h})^{m+1}}{m+1}\,\eta_{j+h} + \frac{(\delta x^{m+1} - (x - x_{j+h+1})^{m+1})}{m+1}\,\eta_{j+h+1}
\end{aligned}$$

Finally, we obtain the following induction formulae:

$$\left\{\begin{aligned}
a_{(i-1)j} &= \sum_h \omega_h \left(a_{i(j-h)(m-1)} - a_{i(j-h-1)(m-1)}\right) \\
q_{(i-1)jm} &= \frac{1}{m}\sum_h \omega_h \left(q_{i(j-h)(m-1)} - q_{i(j-h-1)(m-1)}\right) \qquad m \geq 1 \\
q_{(i-1)j0} &= \sum_h \omega_h \left(e^{\delta x} a_{i(j-h-1)} - a_{i(j-h)} - \frac{1}{m}\sum_{m=1}^{n-i+1} \delta x^m q_{i(j-h-1)(m-1)}\right)
\end{aligned}\right. \qquad (2.3)$$

The initialisation is given by the pay-off:

$$\left.\begin{aligned}
&a_{nj} = q_{nj0} = 0 \qquad j < 0 \\
&\left.\begin{aligned} a_{nj} &= K \\ q_{nj0} &= -K \end{aligned}\right\} \quad j \geq 0
\end{aligned}\right\} \quad \text{for a call}$$

and

$$\left.\begin{aligned}
&a_{nj} = q_{nj0} = 0 \qquad j \geq 0 \\
&\left.\begin{aligned} a_{nj} &= -K \\ q_{nj0} &= K \end{aligned}\right\} \quad j < 0
\end{aligned}\right\} \quad \text{for a put}$$

2.4. Derivatives

Let us set, as usual:

$$\Delta = \frac{\partial \varphi}{\partial S} \qquad \Gamma = \frac{\partial^2 \varphi}{\partial S^2} \qquad \Theta = \frac{\partial \varphi}{\partial t} \qquad \vartheta = \frac{\partial \varphi}{\partial \sigma}$$

The Δ and the Γ are approximated by:

$$\tilde{\Delta}_{t_i}(x) = \frac{a_{ij}\, e^{-x_j}}{K} + \frac{e^{-x}}{K} \sum_{m=1}^{n-i} m\, q_{ijm}\, (x - x_j)^{m-1}$$

$$\tilde{\Gamma}_{t_i}(x) = \frac{e^{-2x}}{K^2} \sum_{m=1}^{n-i} m\, q_{ijm} \left((m-1)\,(x - x_j)^{m-2} - (x - x_j)^{m-1} \right)$$

The approximation $\tilde{\Theta}$ of Θ is obtained thanks to the Black & Scholes PDE:

$$\tilde{\Theta} = r\,\tilde{\varphi} - r\,K\,e^x\,\tilde{\Delta} - \frac{1}{2}\,\sigma^2 K^2 e^{2x}\,\tilde{\Gamma}$$

and the estimation of ϑ by finite difference:

$$\tilde{\vartheta}_t(x) = \frac{1}{2\delta\sigma}\,(\tilde{\varphi}_t(x, \sigma + \delta\sigma) - \tilde{\varphi}_t(x, \sigma - \delta\sigma))$$

the value of $\tilde{\varphi}_t(x, \sigma \pm \delta\sigma)$ being computed with the same δx, by modifying only the coefficients ω_h, so that $\tilde{\vartheta}$ also has a decomposition over the same lattice. A very acceptable precision is obtained with $\delta\sigma = \sigma/100$.

2.5. Remarks

1. Normal diffusion processes (e.g. Heath-Jarrow-Morton model on yields, without volatility smile) are taken into account, with this method, by considering x as an *affine* function of S, instead of a logarithm.

2. One gets the price of a digital option, or a barrier option whose barrier is active only at maturity (e.g. barriers on caps & floors), just by changing the initialisation of the induction.

3. Continuous dividends (i.e. Garman-Kohlhagen model) are easily taken into account by modifying the drift coefficient μ. Discrete ones also, provided their dates belong to the time lattice $(t_i)_{0 \le i \le n}$.

4. This algorithm, well adapted, allows to compute in a very short time the global position and risk analysis of a global portfolio, containing a whole set of European options: the value of each option is computed at the last node of the time lattice before its maturity, even if the latter does not belong to the lattice, and if the strike is not on the x lattice, then one comes back to the prescribed lattice by adjusting the ω_h .

3. Control of the error

In order to find an upper bound for the error on the price and on its derivatives, one has to consider Fourier transform of the functions involved in the algorithm. Let us denote by $\hat{\psi}$ the Fourier transform of a generic function $\psi \in L^1(\mathbf{R})$:

$$\hat{\psi}(s) = \int_{-\infty}^{\infty} \psi(x)\, e^{-isx}\, dx$$

This is a holomorphic function on its domain in the complex plane $\mathbf{C}$.

With the previous notations, the following result holds.

3.1. Theorem:

Assume the option is either a call or a put. Let $p_n = p_{T/n}$ and:

$$\kappa_{n\sigma} = \hat{p}_n^n(-i) - \hat{g}_T(-i)$$

Then, for any $k \in \mathbf{N}$, there exist constants $C_{k\nu}$ depending only on ν and k such that, for any $x \in \mathbf{R}$ and any $n \geq k$, one has:

$$\left|\tilde{\varphi}_n^{(k)}(x) - \varphi^{(k)}(x)\right| \leq K \left(C_{k\nu}\, \sigma\sqrt{T}\, \frac{1 + (\log n)^{\frac{k+\nu-1}{2}}}{n^{\nu/2-1}} + |\kappa_{n\sigma}|\, e^x \right) \quad (3.1)$$

(the second term may be removed for a put). The mean quadratic error has the following upper bound:

$$\sqrt{\int_{-\infty}^{\infty} (\tilde{\varphi}_n(x) - \varphi(x))^2\, g_T(x)\, dx} \leq K C'_\nu\, \sigma\sqrt{T}\, \frac{1 + (\log n)^{\frac{\nu-1}{2}}}{n^{\nu/2-1}}$$

where the constant C'_ν depends only on ν, and $\kappa_{n\sigma}$ satisfies:

$$\kappa_{n\sigma} = \mathcal{O}\left(\sigma^\nu T^{\frac{\nu}{2}} n^{1-\frac{\nu}{2}}\right)$$

This statement is consistent with the previous one when $\nu = 4$.

Remark: In the inequation (3.1) the term $|\kappa_{n\sigma}|\, e^x$ is usually much smaller than the other one. Moreover, one can cancel this term and get a bounded error by modifying the last moment condition on the step function $p_{\delta t}$ and requiring that the scalar product with e^x coincide with that of the Gaussian function $g_{\delta t}$. This modification barely affects the overall result.

Proof. Let H be the Heaviside function:

$$H = \mathbf{1}_{\{x \geq 0\}}$$

For a call, one has:

$$f' = f + K\, H$$

and for a put:

$$f' = f + K\,(H - 1)$$

We keep the notations of theorem 2.1.1. Let:

$$g_n = p_n^{*n} \quad , \qquad g = g_T \qquad \text{and} \qquad q_n = g_n - g$$

The function q_n is fast decreasing at infinity and so are its iterated primitives up to order $\nu - 1$ because the moments of g_n and g coincide up to that order. Let Q_n be the first primitive of q_n which vanishes at infinity:

$$Q_n(x) = \int_{-\infty}^{x} q_n(y)\, dy$$

One has:

$$(f * q_n)' = f * q_n + K\, Q_n$$

both for a call and for a put. Therefore:

$$f * q_n(x) = K\,(\lambda\, e^x - h(x)) \qquad\qquad h(x) = \int_x^{\infty} Q_n(y)\, e^{x-y}\, dy$$

A straightforward computation gives $\lambda = 0$ for a put and $\lambda = \kappa_{n\sigma}$ for a call.

One can easily check that h is of class C^k, fast decreasing at $\pm\infty$, and that:

$$\hat{h}(s) = \frac{\hat{q}_n(s)}{s\,(s+i)} = \frac{\hat{p}_n^n(s) - \hat{g}(s)}{s\,(s+i)}$$

Thus

$$\left|\hat{h}(s)\right| \leq \frac{|\hat{p}_n^n(s) - \hat{g}(s)|}{|s|^2}$$

and one has:

$$\|h^{(k)}\|_{L^\infty} \leq \|s^k \hat{h}\|_{L^1}$$

Let p be the step function p_n when $n = \sigma = T = 1$ and set $\tilde{\sigma} = \sigma\sqrt{T}$. Then:

$$p_n(x) = \sqrt{n}\, p_1(x\sqrt{n}) = \frac{\sqrt{n}}{\tilde{\sigma}}\, p\left(\frac{x\sqrt{n}}{\tilde{\sigma}}\right)$$

$$\hat{p}_n(s) = \hat{p}_1\left(\frac{s}{\sqrt{n}}\right) = \hat{p}\left(\frac{\tilde{\sigma} s}{\sqrt{n}}\right)$$

By assumption, $\hat{p}(s)$ and $\hat{g}$ coincide at 0 up to the order $\nu - 1$, hence:

$$\exists\, M \quad \text{such that} \quad \forall s\,, \quad |s| \leq 1\,, \quad \left|\log \hat{p}(s) + \frac{s^2}{2} - ir\delta t\, s\right| \leq M\, s^\nu$$

and

$$|s| \leq \frac{\sqrt{n}}{\tilde{\sigma}} \Longrightarrow |\hat{p}_n^n(s) - \hat{g}(s)| \leq M\, n^{1-\frac{\nu}{2}} \tilde{\sigma}^\nu\, |s|^\nu \qquad (3.2)$$

When $|s| \geq \dfrac{\sqrt{n}}{\tilde{\sigma}}$ we shall use the following majoration:

$$|\hat{p}_n^n(s) - \hat{g}(s)| \leq |\hat{p}_n^n(s)| + |\hat{g}(s)| = \left|\hat{p}\left(\frac{\tilde{\sigma} s}{\sqrt{n}}\right)\right|^n + e^{-\frac{\tilde{\sigma}^2 s^2}{2}}$$

Let us write:

$$p = \sum_{j=1}^{\nu} \omega_j \chi_j$$

where χ_j is a one-step function whose support has width[3] $2a = \dfrac{\delta x}{\tilde{\sigma}}$ and whose total weight is 1. Then $\sum \omega_j = 1$ and one has:

$$|\chi_j(s)| = \left|\frac{\sin as}{as}\right|$$

[3]One can prove that the positivity condition on te coefficients ω_j implies that a is bounded from above and from below by a positive number.

Therefore:

$$|\hat{p}(s)| \leq \left|\frac{\sin as}{as}\right|$$

It is a well known fact that:

$$|x| \leq \pi \Longrightarrow 0 \leq \frac{\sin x}{x} \leq e^{-\frac{x^2}{6}}$$

We deduce from these inequalities that:

$$\text{If} \quad |s| \leq \frac{\pi\sqrt{n}}{a\tilde{\sigma}} \quad \text{then} \quad |\hat{p}_n(s)|^n \leq e^{-\frac{a^2\tilde{\sigma}^2 s^2}{6}}$$

$$\forall s \in \mathbf{R} \qquad\qquad |\hat{p}_n(s)|^n \leq \left(\frac{\sqrt{n}}{a\tilde{\sigma}s}\right)^n$$

Thus, for any $z \leq \frac{\sqrt{n}}{\tilde{\sigma}}$, one has:

$$\begin{aligned}
\|s^k\hat{h}\|_{L^1} \leq \quad & 2M\, n^{1-\frac{\nu}{2}}\tilde{\sigma}^\nu \int_0^z s^{k+\nu-2}ds \\
+ \quad & 2\int_z^{\frac{\pi\sqrt{n}}{a\tilde{\sigma}}} e^{-\frac{a^2\tilde{\sigma}^2 s^2}{6}} s^{k-2}ds + 2\frac{n^{\frac{n}{2}}}{a^n\tilde{\sigma}^n}\int_{\frac{\pi\sqrt{n}}{a\tilde{\sigma}}}^{\infty} s^{-n+k-2}ds \\
+ \quad & 2\int_z^\infty e^{-\frac{\tilde{\sigma}^2 s^2}{2}} s^{k-2}ds
\end{aligned}$$

Denote these four terms A_1, A_2, A_3, A_4. We have:

$$\begin{aligned}
A_1 &= 2M\,\frac{\tilde{\sigma}^\nu z^{k+\nu-1}}{n^{\frac{\nu}{2}-1}\,(k+\nu-1)} \\
A_3 &= \frac{2\,n^{\frac{k-1}{2}}}{(n-k+1)\,\pi^{n-k+1}a^{k-1}\tilde{\sigma}^{k-1}}
\end{aligned}$$

and one can compute:

$$\int_z^\infty e^{-\frac{s^2}{2}} s^{k-2}ds = \begin{cases} \dfrac{(k-3)!}{2^{\frac{k}{2}-2}\left(\frac{k}{2}-2\right)!}\displaystyle\int_z^\infty e^{-\frac{s^2}{2}}ds & \text{if } k \text{ is even} \\ 2^{\frac{k-3}{2}}\left(\dfrac{k-3}{2}\right)!\, e^{-\frac{z^2}{2}} & \text{if } k \text{ is odd} \end{cases}$$

so that, in any case:

$$\int_z^{\infty} e^{-\frac{s^2}{2}} s^{k-2} ds \le C_k\, e^{-\frac{z^2}{2}}$$

We get:

$$A_2 \le \frac{C'_k}{a^{k-1}\tilde{\sigma}^{k-1}}\, e^{-\frac{a^2\tilde{\sigma}^2 z^2}{6}} \qquad\qquad A_4 \le \frac{C''_k}{\tilde{\sigma}^{k-1}}\, e^{-\frac{\tilde{\sigma}^2 z^2}{2}}$$

Finally, considering a bounded from below by a positive constant depending only on ν (see footnote 3), and denoting $y = \tilde{\sigma} z$, we end in the following majoration of the error, valid for any value of $y \in [0, \sqrt{n}]$:

$$\|h^{(k)}\|_{L^\infty} \le C'''_{\nu k}\, \tilde{\sigma}^{1-k} \left(n^{1-\frac{\nu}{2}}\, y^{k+\nu-1} + e^{-\frac{a^2 y^2}{6}} + \frac{n^{\frac{k-1}{2}}}{\pi^n} \right)$$

Choosing $y = \frac{1}{a}\sqrt{(3\nu - 6)\log n}$ leads to the required upper bound, for:

$$\frac{n^{\frac{k-1}{2}}}{\pi^n} = o\left(n^{1-\frac{\nu}{2}}\right)$$

The mean quadratic error is easily computed from this inequality, and the majoration of $\kappa_{n\sigma}$ is an immediate consequence of (3.2). □

4. Barrier options

4.1. Definition and pricing

As we mentioned in the introduction, one of the main applications of this algorithm is the computation of price and hedge of barrier options. Remember that a *barrier option* is an option which appears, or dies, as soon as the underlying asset goes through (up, or down) a certain limit. Hence, there is 8 kinds of such options: call or put, up or down, in or out. In this section, we consider only these options, that is European options, whose barrier is of "American type" (this means that the barrier is active all along the life of the contract, and not only at maturity[4]). However, our algorithm allows to compute all kinds of barrier options, even with

[4]For options whose barrier is active only at maturity, see the remark at the end of §2.5, remark 2.

non constant or double barrier, "cliquet" or "moving strike" (the strike is equal to the maximum or to the minimum of the underlying asset on the prescribed period of time).

When the volatility and the interest rate are constant, one obtains the price of a barrier option using the "mirror lemma" and Girsanov theorem.

For instance, let S_t be the price of the underlying asset at time t and φ be the price of an up & out call option with maturity T, strike K and barrier $L > K$. Let r be the risk less interest rate, σ the volatility of S_t and set:

$$\lambda = \frac{\mu}{\sigma^2} = \frac{r}{\sigma^2} - \frac{1}{2} \qquad k = \log \frac{K}{S_0} \qquad \ell = \log \frac{L}{S_0}$$

Assume $L \geq S_0$ (otherwise $\varphi = 0$), then:

$$\varphi = \int_k^\ell e^{-rT - \frac{1}{2}\lambda^2\sigma^2 T + \lambda x} \left(S_0 e^x - K\right) \Big(g_{\sigma\sqrt{T}}(x) - g_{\sigma\sqrt{T}}(2\ell - x)\Big)\, dx \tag{4.1}$$

where

$$g_{\sigma\sqrt{T}}(x) = \frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{x^2}{2\sigma^2 T}} \qquad x = \log \frac{S_T}{S_0}$$

4.2. Induction formula

In formula (4.1), one can replace the term $S_0 e^x - K$ by any pay-off, in particular, by the price of the option at time T if this is not the maturity. Therefore, let $\varphi_t(x)$ be the price of the option at time t when $S_t = S_0\, e^x$. One has:

$$\varphi_{t-\delta t}(x) = \varphi^+_{t-\delta t}(x) - \varphi^-_{t-\delta t}(x)$$

with

$$\begin{cases} \varphi^+_{t-\delta t}(x) &= \displaystyle\int_{k-x}^{\ell - x} e^{-r\delta t - \frac{1}{2}\lambda^2\sigma^2\delta t + \lambda y}\, \varphi_t(x+y)\, g_{\sigma\sqrt{\delta t}}(y)\, dy \\ \varphi^-_{t-\delta t}(x) &= \displaystyle\int_{k-x}^{\ell - x} e^{-r\delta t - \frac{1}{2}\lambda^2\sigma^2\delta t + \lambda y}\, \varphi_t(x+y)\, g_{\sigma\sqrt{\delta t}}(2\ell - 2x - y)\, dy \end{cases}$$

Indeed, in this new situation,

$$x = \log \frac{S_{t-\delta t}}{S_0} \qquad \text{and} \qquad y = \log \frac{S_t}{S_{t-\delta t}} = \log \frac{S_t}{S_0} - x \in [k - x\,,\, \ell - x]$$

One can rewrite these integrals with $y = \log(S_t/S_0)$:

$$\begin{cases} \varphi^+_{t-\delta t}(x) &= \displaystyle\int_k^\ell e^{-r\delta t - \frac{1}{2}\lambda^2\sigma^2\delta t + \lambda(y-x)}\, \varphi_t(y)\, g_{\sigma\sqrt{\delta t}}(y-x)\, dy \\ \varphi^-_{t-\delta t}(x) &= \displaystyle\int_k^\ell e^{-r\delta t - \frac{1}{2}\lambda^2\sigma^2\delta t + \lambda(y-x)}\, \varphi_t(y)\, g_{\sigma\sqrt{\delta t}}(2\ell - x - y)\, dy \end{cases}$$

or, still shifting the Gaussian function and absorbing the exponential factor:

$$\begin{cases} \varphi^+_{t-\delta t}(x) &= e^{-r\delta t} \displaystyle\int_k^\ell \varphi_t(y)\, g_{\sigma\sqrt{\delta t}}(y - x - \mu\,\delta t)\, dy \\ \varphi^-_{t-\delta t}(x) &= e^{-r\delta t + 2\lambda(\ell - x)} \displaystyle\int_k^\ell \varphi_t(y)\, g_{\sigma\sqrt{\delta t}}(y + x - 2\ell - \mu\,\delta t)\, dy \end{cases} \tag{4.2}$$

Assume now that the function φ_t is of the form:

$$\varphi_t(x) = e^{2\lambda(\ell - x)}\psi_t(x)$$

then:

$$\begin{cases} \varphi^+_{t-\delta t}(x) &= e^{-r\delta t + 2\lambda(\ell - x)} \displaystyle\int_k^\ell \psi_t(y)\, g_{\sigma\sqrt{\delta t}}(y - x + \mu\,\delta t)\, dy \\ \varphi^-_{t-\delta t}(x) &= e^{-r\delta t} \displaystyle\int_k^\ell \psi_t(y)\, g_{\sigma\sqrt{\delta t}}(y + x - 2\ell + \mu\,\delta t)\, dy \end{cases} \tag{4.3}$$

This remark will be of importance in the next section.

4.3. Convolution algorithm

The idea is now to replace the Gaussian functions by step functions with the same first three moments, and whose steps respect the meshing in x and y. We then obtain, as in §2, a sequence of functions $\tilde\varphi_{T-i\delta t}$ which approximate $\varphi_{T-i\delta t}$.

However, the expression of $\tilde\varphi^-_{t-\delta t}$ is not anymore given by a simple primitive of $\tilde\varphi_t$, but it is multiplied by an exponential in x. Suppose that $\tilde\varphi_T$ is a combination of a constant and e^x. Then $\tilde\varphi^+_{T-\delta t}$ is, as previously, a combination of e^x and a polynomial of degree 1, but $\tilde\varphi^-_{T-\delta t}$ is a combination of $e^{-(1+2\lambda)x}$ and $e^{-2\lambda x}\times$ *polynomial of degree 1*. Thus $\tilde\varphi_{T-i\delta t}$ is decomposed into a combination of e^x and a polynomial of degree i on the one hand, and, on the other hand, the same kind of combination with e^{-x} instead of e^x, multiplied by $e^{-2\lambda x}$.

We shall now assume that $\tilde{\varphi}_{t_i}(x)$ has the following split shape when $x \in [x_j, x_{j+1}]$:

$$\tilde{\varphi}_{t_i}(x) = a_{ij} e^{x-x_j} + \sum_{m=0}^{n-i} q_{ijm}\,(x-x_j)^m + e^{2\lambda(\ell-x)} \left(b_{ij} e^{x_{j+1}-x} + \sum_{m=0}^{n-i} p_{ijm}\,(x_{j+1}-x)^m \right) \tag{4.4}$$

In the induction formulae (4.2) and (4.3), we shall therefore replace the exact Gaussian function $g_{\sigma\sqrt{\delta t}}$ by a ν-steps function $p_{\delta t}$ with the same first $\nu-1$ moments (see §2.2) and, as we shall see, get an induction on the coefficients a_{ij} , b_{ij} , q_{ijm} and p_{ijm} . For this purpose, and keeping the notations of §2.2, one has to compute the integrals:

$$\begin{cases} I^{++} = \displaystyle\int_k^\ell e^{y-x_j}\, \eta_j(y)\, \eta_h(y-x)\, dy \\ I^{+-} = \displaystyle\int_k^\ell e^{y-x_j}\, \eta_j(y)\, \eta_h(y+x-2\ell)\, dy \end{cases}$$

$$\begin{cases} I^{-+} = \displaystyle\int_k^\ell e^{x_{j+1}-y}\, \eta_j(y)\, \eta_h(y-x)\, dy \\ I^{--} = \displaystyle\int_k^\ell e^{x_{j+1}-y}\, \eta_j(y)\, \eta_h(y+x-2\ell)\, dy \end{cases}$$

and also:

$$\begin{cases} J_m^{++} = \displaystyle\int_k^\ell (y-x_j)^m\, \eta_j(y)\, \eta_h(y-x)\, dy \\ J_m^{+-} = \displaystyle\int_k^\ell (y-x_j)^m\, \eta_j(y)\, \eta_h(y+x-2\ell)\, dy \end{cases}$$

$$\begin{cases} J_m^{-+} = \displaystyle\int_k^\ell (x_{j+1}-y)^m\, \eta_j(y)\, \eta_h(y-x)\, dy \\ J_m^{--} = \displaystyle\int_k^\ell (x_{j+1}-y)^m\, \eta_j(y)\, \eta_h(y+x-2\ell)\, dy \end{cases}$$

One easily checks that all these integrals can be expressed in terms of:

$e^{x-x_{j'}}$	for I^{++} and I^{--}
$e^{x_{j'+1}-x}$	for I^{+-} and I^{-+}
$(x-x_{j'})^{m+1}$	for J^{++} and J^{--}
$(x_{j'+1}-x)^{m+1}$	for J^{+-} and J^{-+}

for suitable values of j', and of constant terms. Like in the case of standard European options, the shift $\mu\delta t$ (in both directions) is taken into account through the coefficients ω_h, because the step function $p_{\delta t}$ is required to have the same moments as the shifted Gaussian function $g_{\sigma\sqrt{\delta t}}(x \pm \mu\delta t)$.

4.4. Non constant interest rate and volatility

We now assume that the volatility and the interest rate vary from one time step to another. Let σ_i and r_i be those of the mesh $[t_i, t_{i+1}]$, and λ_i and μ_i be the corresponding value of λ and μ. The structure of (4.4) is no more preserved: the integrals I^{-+} and J_m^{-+} keep the former λ_i, while I^{+-} and J_m^{+-} get the new λ_{i-1}. As well, the integrals I^{++} and J_m^{++} keep the same shape, but I^{--} and J_m^{--} get a factor $e^{2(\lambda_i-\lambda_{i-1})(\ell-x)}$. One gets, in the end, a decomposition of $\tilde{\varphi}_i(x)$, $x \in [x_j, x_{j+1}]$, which also splits into two parts, a "direct" one and a "mirror" one:

$$\tilde{\varphi}_i(x) = \sum_k a_{ijk}\, e^{\alpha_k(x-x_j)} + Q_{ij}(x) + e^{2\lambda(\ell-x)} \left(\sum_k b_{ijk}\, e^{\beta_k(x-x_j)} + P_{ij}(x) \right)$$

where λ is a mean value of the λ_i's and the exponents α_k and β_k are linked to the differences $\lambda_{i-1} - \lambda_i$.

An important issue, in terms of complexity, is the number of such exponents. At each step, the set formed by them is enriched with respect to $\lambda_i - \lambda$. One sees that if $\lambda_i = \lambda_{i-1}$, then this set is stable, and the number of coefficients increases by its cardinal (the degree of Q_{ij} and P_{ij} increases by 1). In the contrary, the set is doubled, but the degree of the polynomials remains unchanged. The number of meshes in x being $\nu\,(n-i+1)$, the maximum number of coefficients at time t_i is $\nu\,(n-i+1)\,2^{n-i+1}$. This could seem frightening, but we shall see in the next section that 5 steps are enough to get the precision of a 300 steps binomial tree.

Moreover, one can reasonably expect that the λ_i's take at most 3 or 4 different values, so that the total number of coefficients remains under 1000.

For even more efficiency, one can approximate the exponential functions $e^{\alpha_k x}$ and $e^{\beta_k x}$ by "splines", that is polynomials with the same degree as Q_{ij} and P_{ij} which have a maximal order contact at the nodes of the lattice. In this case, the structure of the decomposition (4.4) is preserved and the error keeps the same order of magnitude.

4.5. Performances achieved

The computing time is theoretically linked to the number of coefficients to compute. Even in the case of varying λ, it should not exceed that of a 50 steps binomial tree after 5 steps with $\nu = 6$, for the binomial tree must compute $50 \times 51/2 = 1275$ coefficients, which is more than a 5 steps convolution method.

As we said, when $\nu = 6$, the precision after 5 steps is 0,5% on prices, Δ and Γ. It reaches 0,1% after 7 steps and a little better after 10 steps.

For European options, we get the same precisions with $\nu = 4$. When $\nu = 6$, the precision is about 10 times better than for the barrier options with the same number of steps.

References

[1] J. Cox, S. Ross and M. Rubinstein *Option Pricing: A Simplified Approach* Journal of Financial Economics, 7, Sept. 1979, 229-263.

[2] J. Cox and M. Rubinstein *Option Markets* Prentice Hall Ed., Englewood Cliffs N.J., 1985.

[3] G. Barone-Adesi and R. Whaley *Efficient analytic approximation of American option value* Journal of Finance, 42, June 1987, 301-320.

[4] L. McMillan *Analytic approximation for the American put option* Journal of Finance, 39, Dec. 1984, 1511-1524.