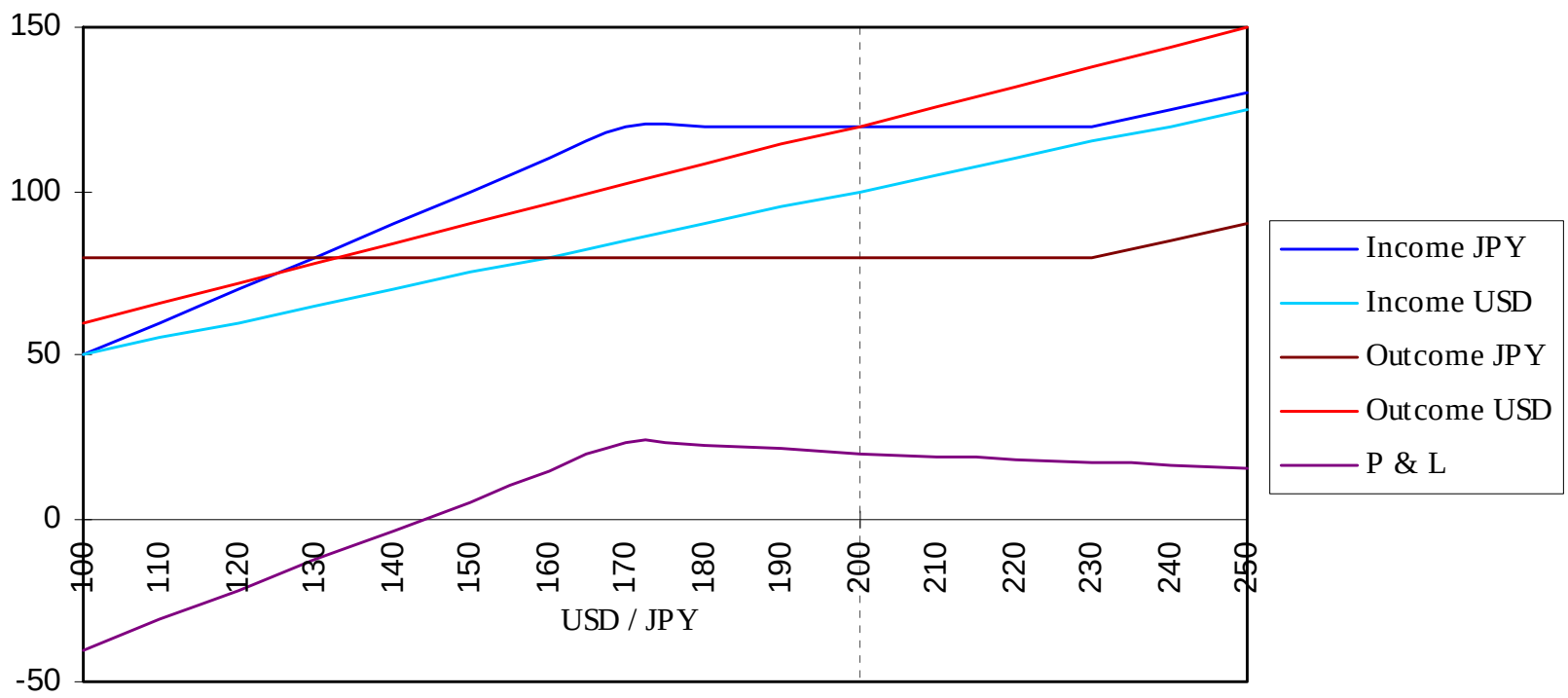
Challenges in Math. Finance

- Market Economy requires the use of Derivatives to fit the Non-linear Behavior of the Capital Markets
- Often Path-dependent
  - Exotic Options
  - Structured Products

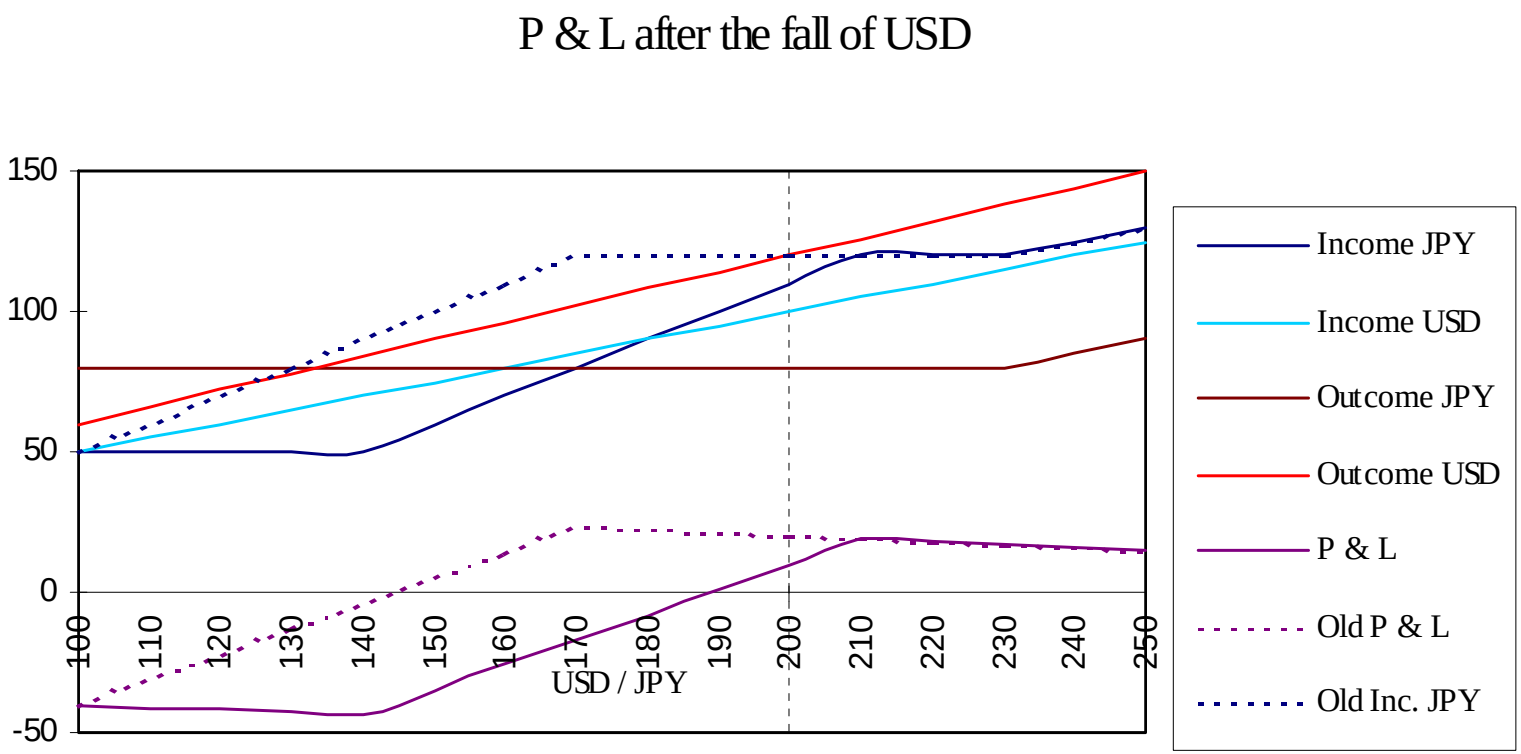
Challenges in Math. Finance

- The *Japan Air Lines* Problem
  - Linear Hedge to match Income and Outcome
  - Option to fit big Market Moves and Inference on Prices
  - Path Dependency : What if USD/JPY comes back ?

Challenges in Math. Finance



Challenges in Math. Finance



Challenges in Math. Finance

Industry	**Bank**
Price of Good	*Price of Service*
Production Costs	*Cost of Hedging + Price of Risk*

- From Sales : No Link between Prices and Costs
- How Math's can help to Structure Prices

Challenges in Math. Finance

- Liquidity = Possibility to Unwind the Position

Small Volume	✉	*Low Volatility* *High Bid/Offer*
Big Volume	✉	*High Volatility* *Low Bid/Offer*

- "A la Heisenberg" inequality :

$$Volume \times (Sell - Buy) \geq Trading\ Costs$$

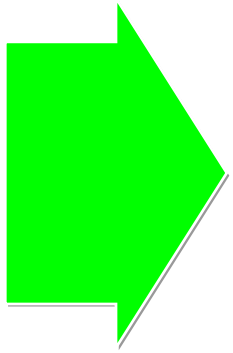
- When a figure is traded, it becomes volatile

Challenges in Math. Finance

Statistics (+ Economical Analysis)

Probabilities

Stochastic Control



Market

Practical

Device

~ P.D.E.

Complexity (Numerical Techniques)

4 "Quantum like" Oppositions

- Discrete Time Finance *vs.* Continuous Time Finance
- Parametric Statistic *vs.* Non Parametric Statistics
- Bid / Offer *vs.* Tick Price Modeling
- Arbitrage Theory *vs.* Equilibrium Theory

Challenges in Math. Finance

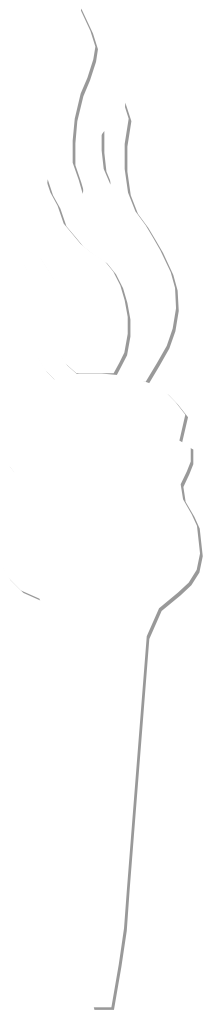
Discrete Time Finance *vs.* Continuous Time Finance

- Gaye : *No Price without a Hedge*
- Taleb : *Continuous Time Pricing*
  Discrete Time Hedging

The Complexity Paradox :

Continuity is a economic approximation of Discreteness

Challenges in Math. Finance



Discrete Time Finance *vs.* Continuous Time Finance

- Buy / Sell and Hedge are Static Decisions
- Every price is computed in prevision of an at least partially dynamic hedging strategy
  - Take a view on market behavior in the future
  - Hedging means exploiting instantaneous correlations

Challenges in Math. Finance

Parametric Statistics *vs.* Non Parametric Statistics

- The number of data is at the same time large and small
  - "Inter-scale" context
- Data indexed by almost continuous parameters (interest rates, implied volatilities, etc.)
  - The optimal modeling depends on the aim of the model

Challenges in Math. Finance

Parametric Statistics *vs.* Non Parametric Statistics

- Can we evaluate the risk left out by a model ? Is it reliable ?
  - Statistics *vs.* Econometrics (+ knowledge of market behavior)
- Should we hedge totally according to a model, knowing that there is an off-model risk ?
  - What is the price of this risk ?

Bid / Offer *vs.* Tick Price Modeling

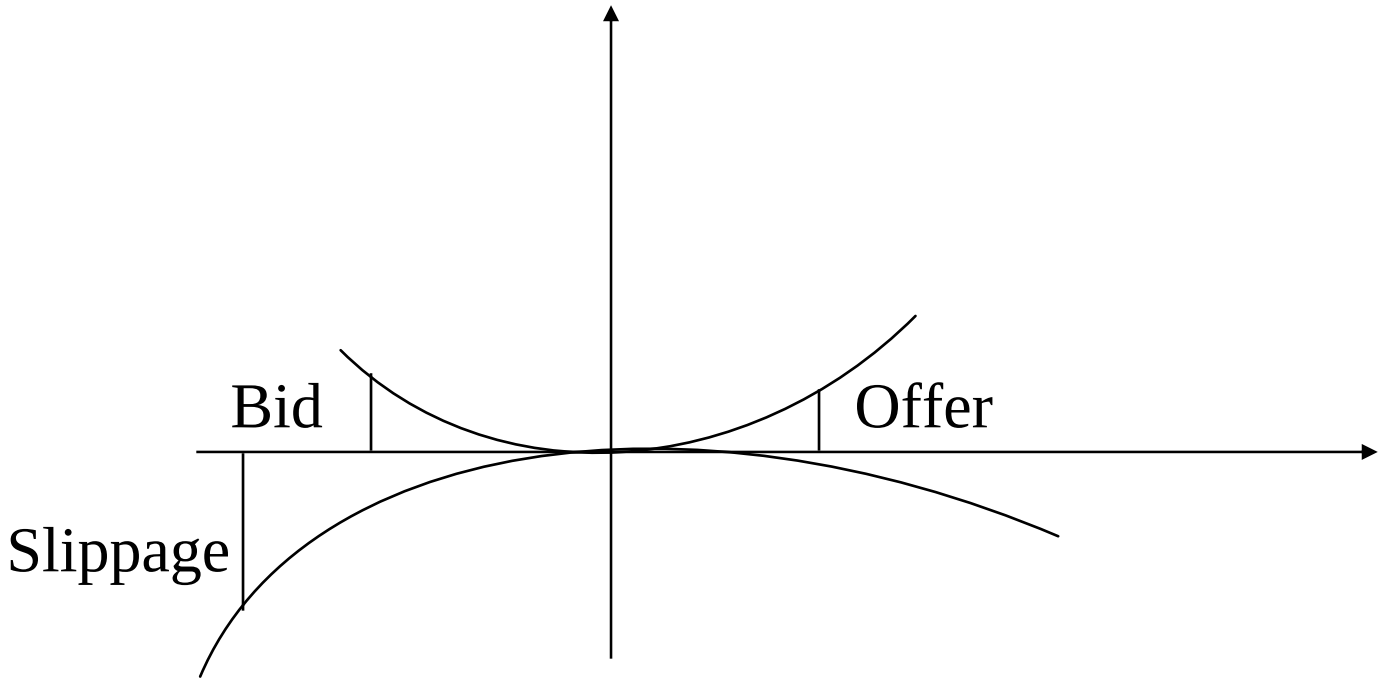
or, Do Proportional Transaction Costs Exist ?

- Taleb : *No*
- Gaye : *Yes*

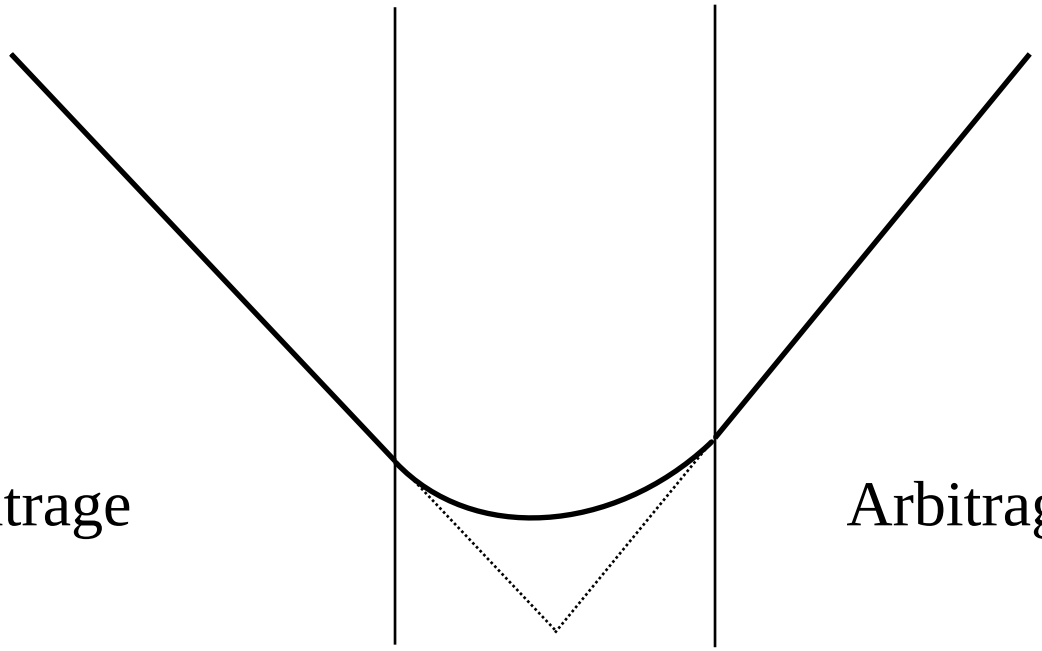
Question : Are we Long or Short Gamma ?

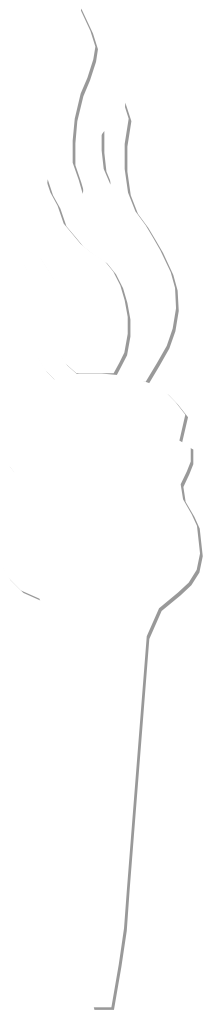
- Long Gamma : What is the optimal bid to maximise the Theta ?
- Short Gamma : How fast do we run (what slippage) ?

Challenges in Math. Finance



Arbitrage Theory *vs.* Equilibrium Theory





Margins and the Pricing of Risk

- No Arbitrage = No Business
- Are margins to cover current expenses or to price an off-model risk ?
- Only one measure of risk : its price on the market
- Are models with a "low level of arbitrage" consistent ?

A principle in finance : No Infinite Number

High, low, big, small mean finite numbers.

- ↗ Robustness should be systematically examined

Quasi-Arbitrage and Risk Evaluation

- How to quantify the difference between a theoretical arbitrage and a real arbitrage opportunity, that is its resistance to model inadequacy (transaction costs, uncertainty of parameters) ?
- Does one have to pay the total amount of the arbitrage to cancel the off-model risk ?
- Can the mathematician provide an optimal family of hedging ("replication") strategies compatible with market prices, from the most risk averse to the most bullish, but cheapest and most simple ?

Value at Risk (VAR)

- How many variables, which ones ?
  - ↗ Galerkin approximations
- Non linear VAR problem :

$f : \mathbb{R}^N$ or *Wiener Space* $\rightarrow \mathbb{R}^N$, $N \geq 100$

f is highly not convex and each evaluation requires several seconds (whole bank portfolio)

E = Ellipsoid of uncertainty $\subset \mathbb{R}^N$ or *Wiener Space*

Problem : Evaluate $\min_E f$ in a reasonable time

Value at Risk (cont'd)

- Static or Dynamic VAR ?
  - Can dynamic hedging be, in certain cases, more risky than no hedge at all ?
  - How can one forecast liquidity holes ?
  - Would it be possible to define an index of market instability (catastrophe theory, bifurcation theory, Lyapunov exponents, else) ?
- Effect of regulations on market dynamics
- What is the optimal hedge close to an option singularity ? What inference on prices ?

Complexity

- The complexity of any computation depends on the dimension of the smallest Markov model that contains the initial model
  - Non Markov models can hide a high complexity
- Exponential with respect to dimension (except Monte-Carlo and related approaches)
- Find a numerical solution to the *N*-dimensional Hamilton-Jacobi-Bellman equation
  - What simplifications
  - Example : one underlying with stochastic volatility

Off-model Risk, Over-hedge and Optimal Hedging

- Define sequences of embedded models, not to get a perfect model, but to measure off-model risk
- Can one theoretically bound the off-model risk ?
- How can over-hedged strategies fit the market ?
  - Optimal sharing between static and dynamic hedging
  - Find smooth / PL and meaningful penalty functions
  - Numerical implementation of high dimensional Lagrangian optimisation
- Example : Optimal hedge of a book of barrier options

Optimal Decision and Free Boundary Problems

- Almost any problem of optimal decision making ends into a PDE with a free boundary
  - Efficient numerical method to compute the free boundary in several dimensions
  - Example : American option with stochastic volatility and interest rates
- Boundary conditions are usually non smooth (kinks, steps)
  - Regularity of solutions and speed of algorithms

A Non-Linear GARCH Model (Taleb)

S_t = spot value of underlying

σ_t = implied volatility

$$S_{t+1} = S_t \, (1 + \sigma_t \xi_t \,)$$

$$\sigma_{t+1} = f(\sigma_t \, , S_t \, ,..., S_{t-\alpha} \,)$$

- f is piecewise linear (PL) $\alpha = 2$ or 3
  - Example : Volatility goes up if spot falls once, but also if it drops fast and repeatedly, etc.
- What is the ergodic limit of such a process ?

Conclusion

- Traders often think academics barely answer their real questions
- 90% of problems are trivial, or almost, from the theoretical point of view
- 9 of remaining 10% are unsolvable problems
  - Difficult for an academic to find "interesting" as well as "useful" results
- It is a phantasm to believe that math's can answer everything
  - Problem : What is the price of this phantasm ?