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ON PROBABILITY CHARACTERISTICS OF "DOWNFALLS" IN A STANDARD BROWNIAN MOTION*

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(*Translated by M. V. Khatuntseva*)

Abstract. For a Brownian motion $B=(B_t)_{t\leqq 1}$ with $B_0=0$, $\mathbf{E}B_t=0$, $\mathbf{E}B_t^2=t$ problems of probability distributions and their characteristics are considered for the variables

$$\mathbb{D}=\sup_{0\leqq t\leqq t'\leqq 1}(B_t-B_{t'}),\qquad \mathbb{D}_1=B_\sigma-\inf_{\sigma\leqq t'\leqq 1}B_{t'},$$
$$\mathbb{D}_2=\sup_{0\leqq t\leqq \sigma'}B_t-B_{\sigma'},$$

where σ and σ' are times (non-Markov) of the absolute maximum and absolute minimum of the Brownian motion on $[0,1]$ (i.e., $B_\sigma=\sup_{0\leqq t\leqq 1}B_t$, $B_{\sigma'}=\inf_{0\leqq t'\leqq 1}B_{t'}$).

Key words. Brownian motion, "downfalls" and "range," Lévy theorem, Brownian meander

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1. Introduction. Statement of the results.

1. A standard Brownian motion $B=(B_t)_{0\leqq t\leqq 1}$ (with $B_0=0$, $\mathbf{E}B_t=0$, $\mathbf{E}B_t^2=t$) is considered on a probability space $(\Omega,\mathcal{F},\mathbf{P})$.

Our main interest is to study the probability properties of the values

$$\mathbb{D}=\sup_{0\leqq t\leqq t'\leqq 1}(B_t-B_{t'}),\tag{1}$$

$$\mathbb{D}_1=B_\sigma-\inf_{\sigma\leqq t'\leqq 1}B_{t'},\tag{2}$$

$$\mathbb{D}_2=\sup_{0\leqq t\leqq \sigma'}B_t-B_{\sigma'},\tag{3}$$

where σ and σ' are defined ($\mathbf{P}$-a.s. uniquely) by the relations

$$B_\sigma=\sup_{0\leqq t\leqq 1}B_t,\tag{4}$$

$$B_{\sigma'}=\inf_{0\leqq t'\leqq 1}B_{t'}.\tag{5}$$

From the given definitions it is clear that the value $\mathbb{D}$ characterizes the *maximal* possible "*downfall*" in trajectories of the Brownian motion on the time interval $[0,1]$.

The value $\mathbb{D}_1$ shows how the trajectory of the Brownian motion can "downfall" from *absolute* (on $[0,1]$) *maximum* B_σ to (partial) *minimum* $\inf_{\sigma\leqq t'\leqq 1}B_{t'}$. Respectively, $\mathbb{D}_2$ characterizes the "downfall" from (partial) *maximum* $\sup_{0\leqq t\leqq\sigma'}B_t$ to *absolute* (on $[0,1]$) *minimum* $B_{\sigma'}$.

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It is useful to note that σ and σ' are *non-Markov*.

2. It is clear that

$$\mathbb{D} = \begin{cases} \sup\limits_{0\leqq t\leqq 1} \Big(B_t - \inf\limits_{t\leqq t'\leqq 1} B_{t'}\Big) \geqq \mathbb{D}_1, \\ \sup\limits_{0\leqq t'\leqq 1} \Big(\sup\limits_{0\leqq t\leqq t'} B_t - B_{t'}\Big) \geqq \mathbb{D}_2. \end{cases}$$

So

$$\max(\mathbb{D}_1,\, \mathbb{D}_2) \leqq \mathbb{D}. \tag{6}$$

If we denote by

$$\mathbb{R} = \sup_{0\leqq t\leqq 1} B_t - \inf_{0\leqq t\leqq 1} B_t \tag{7}$$

the "*range*" of the Brownian motion, then we find that

$$\mathbb{R} = \sup_{0\leqq t,t'\leqq 1} (B_t - B_{t'})$$

and

$$\max(\mathbb{D}_1, \mathbb{D}_2) \leqq \mathbb{D} \leqq \mathbb{R}. \tag{8}$$

It is well known that the consideration of the statistic $\mathbb{R}$ is useful in a posteriori analysis of the behavior of the trajectories of the process B on the time interval $[0, 1]$. At the same time, the statistics $\mathbb{D}, \mathbb{D}_1, \mathbb{D}_2$ give more information on the *current* behavior of the trajectories which is especially important when the observations need to be ranked by time.

3. Passing to the statement of the results concerning the probability distributions and their characteristics for statistics $\mathbb{D}, \mathbb{D}_1$, and $\mathbb{D}_2$, first of all we note that $\mathbb{D}_1$ and $\mathbb{D}_2$ coincide in distribution:

$$\mathbb{D}_1 \overset{\text{law}}{=} \mathbb{D}_2. \tag{9}$$

(It easily follows from the fact that the process $\widehat{B} = (\widehat{B}_t)_{0\leqq t\leqq 1}$ with $\widehat{B}_t = B_1 - B_{1-t}$ is also a Brownian motion.)

Theorem 1. *For the standard Brownian motion*

$$\mathbb{D} \overset{\text{law}}{=} \sup_{0\leqq t\leqq 1} |B_t|. \tag{10}$$

The mathematical expectation

$$\mathbf{E}\,\mathbb{D} = \sqrt{\frac{\pi}{2}} \quad (= 1.2533\ldots) \tag{11}$$

and the distribution function $F_{\mathbb{D}}(x) = \mathbf{P}\{\mathbb{D} \leqq x\}$ *is given by the relation*

$$F_{\mathbb{D}}(x) = 1 - \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \int_{-x}^{x} \left[e^{-(y+4kx)^2/2} - e^{-(y+2x+4kx)^2/2}\right] dy. \tag{12}$$

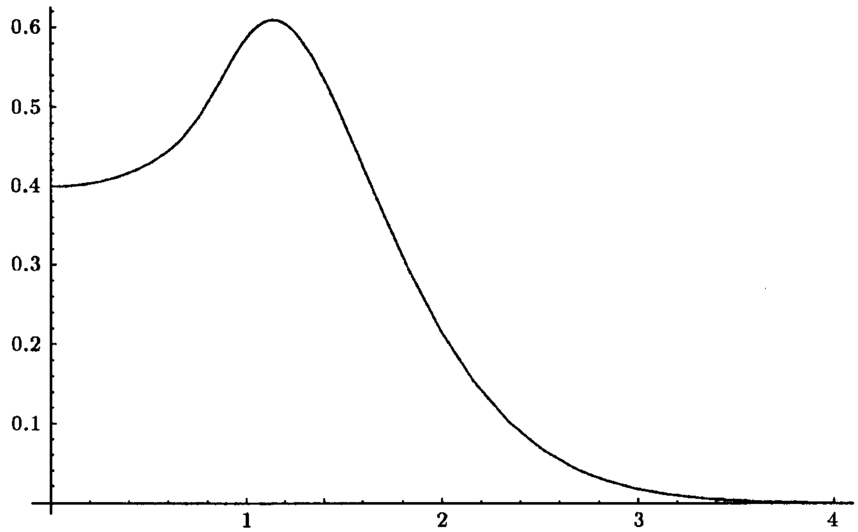

FIG. 1. *The function* $f_{\mathbb{D}_1}(x) = \sqrt{8/\pi}\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-k^2x^2/2}$, $x > 0$.

THEOREM 2. *For the standard Brownian motion*

$$\mathbb{D}_1 \overset{\text{law}}{=} \sup_{g \leqq t \leqq 1} |B_t|, \tag{13}$$

where

$$g = \sup\{t \leqq 1\colon B_t = 0\} \tag{14}$$

is the time of the last zero of the process $B = (B_t)_{0 \leqq t \leqq 1}$.

The expectation

$$\mathbf{E}\,\mathbb{D}_1 = \sqrt{\frac{8}{\pi}}\log 2 \quad (= 1.1061\ldots) \tag{15}$$

and the distribution function $F_{\mathbb{D}_1}(x) = \mathbf{P}\{\mathbb{D}_1 \leqq x\}$ *has a density (Figure* 1*)*

$$f_{\mathbb{D}_1}(x) = \sqrt{\frac{8}{\pi}}\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-k^2x^2/2}, \quad x > 0. \tag{16}$$

4. Series (16) is very similar to the series which describes the distribution probability $F_{\mathbb{K}}(x)$ in *the goodness-of-fit Kolmogorov test* and the probability distribution $F_{\mathbb{R}}(x)$ of the statistic of the *"range"* $\mathbb{R}$.

Namely, the distribution function

$$F_{\mathbb{K}}(x) = \mathbf{P}\Big\{\sup_{0 \leqq t \leqq 1}\big|b(t)\big| \leqq x\Big\}, \tag{17}$$

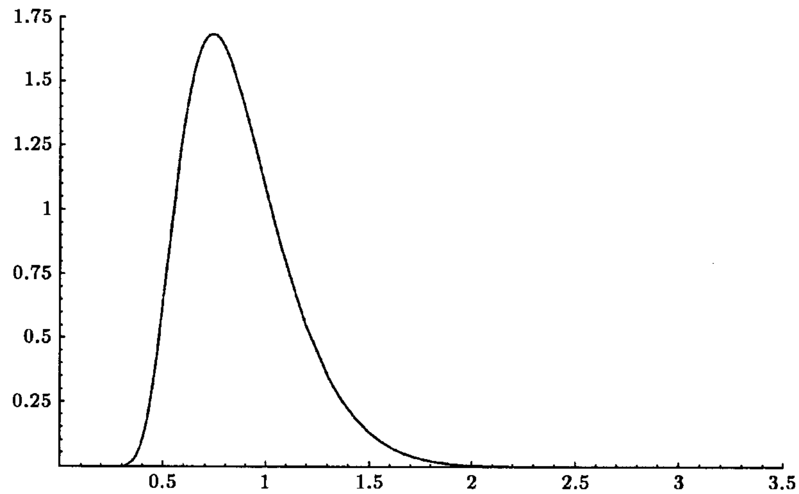

FIG. 2. *The function* $f_{\mathbb{R}}(x) = 8/\sqrt{2\pi}\sum_{k=1}^{\infty}(-1)^{k-1}k^2e^{-k^2x^2/2}$, $x > 0$.

where $b = (b(t))_{0\leqq t\leqq 1}$ is a Brownian bridge $((b(t);\ t \leqq 1) \overset{\text{law}}{=} (B_t - tB_1;\ t \leqq 1))$, is given (see [7], [5], and [4]) by the formula

$$F_{\mathbb{K}}(x) = 1 - 2\sum_{k=1}^{\infty}(-1)^{k-1}e^{-2k^2x^2}, \quad x > 0, \tag{18}$$

or, equivalently, by

$$F_{\mathbb{K}}(x) = \frac{\sqrt{2\pi}}{x}\sum_{k=1}^{\infty}e^{-(2k-1)^2\pi^2/x^2}, \quad x > 0.$$

It is also known (see, for example, [6]), that the probability distribution $F_{\mathbb{R}}(x) = \mathbf{P}\{\mathbb{R} \leqq x\}$ has the density (Figure 2)

$$f_{\mathbb{R}}(x) = \frac{8}{\sqrt{2\pi}}\sum_{k=1}^{\infty}(-1)^{k-1}k^2e^{-k^2x^2/2}, \quad x > 0. \tag{19}$$

From (18) and (19) it follows that the density $f_{\mathbb{K}}(x) = F'_{\mathbb{K}}(x)$ is defined by (Figure 3)

$$f_{\mathbb{K}}(x) = 8x\sum_{k=1}^{\infty}(-1)^{k-1}k^2e^{-2k^2x^2}, \quad x > 0,$$

and, in addition,

$$f_{\mathbb{R}}(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{x}\,f_{\mathbb{K}}\left(\frac{x}{2}\right).$$

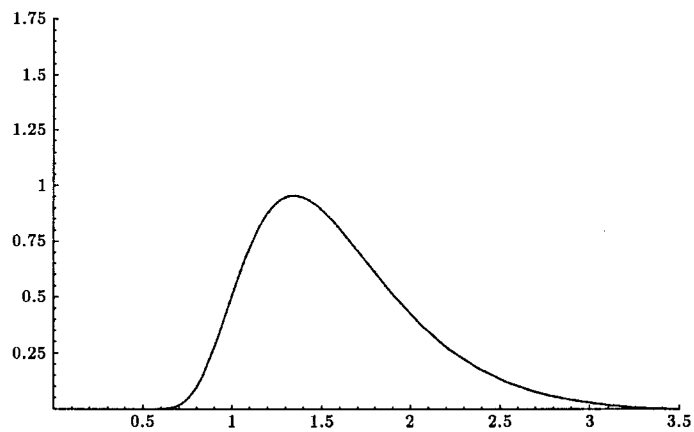

FIG. 3. *The function* $f_{\mathbb{K}}(x) = 8\sum_{k=1}^{\infty}(-1)^{k-1}k^2 x e^{-2k^2x^2}$, $x > 0$.

From here we specifically conclude that

$$\mathbf{E}\,\mathbb{R} = \sqrt{\frac{8}{\pi}} \quad (= 1.5957\ldots)$$

(cf. (8), (11), (15)).

2. Proof of Theorem 1.

1. If it is stated that $\mathbb{D} \overset{\text{law}}{=} \sup_{0\leqq t\leqq 1}|B_t|$ (property (10)), then (12) follows from the known results concerning the distribution $\mathbf{P}\{\sup_{0\leqq t\leqq 1}|B_t| \leqq x\}$. (See, for example, [1, p. 250, 1.1.4].)

The proof of the needed property (10) is very simple. Let

$$M_t = \sup_{0\leqq u\leqq t} B_u$$

and let L_t be a local time of the Brownian motion B on $[0,t]$:

$$L_t = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I\big(|B_s| \leqq \varepsilon\big)\,ds.$$

By one of the fundamental properties of a Brownian motion,

$$(M_t - B_t,\, M_t;\ t \leqq 1) \overset{\text{law}}{=} \big(|B_t|,\, L_t;\ t \leqq 1\big) \tag{20}$$

("*Lévy theorem*" [9, p. 230]).

So

$$\mathbb{D} = \sup_{0\leqq t\leqq t'\leqq 1}(B_t - B_{t'}) = \sup_{0\leqq t'\leqq 1}\Big(\sup_{0\leqq t\leqq t'} B_t - B_{t'}\Big)$$
$$= \sup_{0\leqq t'\leqq 1}(M_{t'} - B_{t'}\Big) \overset{\text{law}}{=} \sup_{0\leqq t\leqq 1}|B_t|.$$

2. The relation $\mathbf{E}\,\mathbb{D} = \sqrt{\pi/2}$ is widely known and may be obtained from (12).

To complete the presentation we give two derivations which are not directly connected with (12) for the distribution $F_{\mathbb{D}}(x) = \mathbf{P}\{\mathbb{D} \leqq x\}$.

Let $\beta = (\beta_t)_{t \geqq 0}$ be a Brownian motion, $S_1 = \inf\{t \geqq 0\colon |\beta_t| = 1\}$. If we use the following self-similarity property: for $a > 0$,

$$(\beta_{at};\ t \geqq 0) \overset{\text{law}}{=} (a^{1/2}\beta_t;\ t \geqq 0),$$

then we obtain that, for $x > 0$,

$$\begin{aligned}\mathbf{P}\Big\{\sup_{0 \leqq t \leqq 1} |\beta_t| \leqq x\Big\} &= \mathbf{P}\Big\{\sup_{0 \leqq t \leqq 1} |\beta_{t/x^2}| \leqq 1\Big\} = \mathbf{P}\Big\{\sup_{0 \leqq t \leqq 1/x^2} |\beta_t| \leqq 1\Big\} \\ &= \mathbf{P}\Big\{S_1 \geqq \frac{1}{x^2}\Big\} = \mathbf{P}\Big\{\frac{1}{\sqrt{S_1}} \leqq x\Big\}.\end{aligned}$$

In other words,

$$\sup_{0 \leqq t \leqq 1} |\beta_t| \overset{\text{law}}{=} \frac{1}{\sqrt{S_1}}.$$

From properties of the normal distribution it follows that for $\sigma > 0$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/(2\sigma^2)}\, dx = \sigma.$$

So

$$\mathbf{E}\,\mathbb{D} = \mathbf{E} \sup_{0 \leqq t \leqq 1} |\beta_t| = \mathbf{E}\,\frac{1}{\sqrt{S_1}} = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathbf{E}\, e^{-x^2 S_1/2}\, dx.$$

It is known that for S_1 (see, for example, [9, p. 68] and [10, p. 303]) the Laplace transform

$$\mathbf{E}\, e^{-\lambda S_1} = \frac{1}{\cosh \sqrt{2\lambda}}.$$

Thus

$$\begin{aligned}\mathbf{E}\,\mathbb{D} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dx}{\cosh x} = 2\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^x\, dx}{e^{2x} + 1} = 2\sqrt{\frac{2}{\pi}} \int_1^\infty \frac{dy}{1 + y^2} \\ &= 2\sqrt{\frac{2}{\pi}} \arctan x \,\Big|_1^\infty = 2\sqrt{\frac{2}{\pi}}\,\frac{\pi}{4} = \sqrt{\frac{\pi}{2}}.\end{aligned}$$

The second derivation of the relation $\mathbf{E}\,\mathbb{D} = \sqrt{\pi/2}$ is based on the property

$$\sup_{0 \leqq t \leqq 1} |B_t| \overset{\text{law}}{=} \frac{1}{2} \int_0^1 \frac{ds}{R_s^{(2)}} \tag{21}$$

(see [9, p. 250]), where $(R_s^{(2)})_{s \leqq 1}$ is the Bessel process of order 2, for which

$$R_s^{(2)} = \widehat{\beta}_s + \frac{1}{2} \int_0^s \frac{du}{R_u^{(2)}}, \tag{22}$$

where $\widehat{\beta} = (\widehat{\beta}_s)_{s \leqq 1}$ is some Brownian motion.

From (21) and (22), we immediately see that

$$\mathbf{E}\,\mathbb{D} = \mathbf{E} \sup_{0 \leqq t \leqq 1} |\beta_t| = \mathbf{E}\, R_1^{(2)} = \mathbf{E}\sqrt{\xi_1^2 + \xi_2^2},$$

where ξ_1 and ξ_2 are independent standard Gaussian variables with parameters 0 and 1. Since $\mathbf{E}\sqrt{\xi_1^2 + \xi_2^2} = \sqrt{\pi/2}$, we arrive at the needed relation $\mathbf{E}\,\mathbb{D} = \sqrt{\pi/2}$ again.

3. Proof of Theorem 2.

1. From the Lévy theorem (20) it follows that

$$(M_t - B_t,\, M_t,\, B_t;\ t \leqq 1) \overset{\text{law}}{=} \big(|B_t|,\, L_t,\, L_t - |B_t|;\ t \leqq 1\big).$$

From here we can conclude that

$$(M_t - B_t,\, M_t,\, B_t;\ \sigma \leqq t \leqq 1) \overset{\text{law}}{=} \big(|B_t|,\, L_t,\, L_t - |B_t|;\ g \leqq t \leqq 1\big).$$

So

$$\Big(B_\sigma,\ \sup_{\sigma \leqq t \leqq 1} (M_t - B_t - M_t)\Big) \overset{\text{law}}{=} \Big(L_g - |B_g|,\ \sup_{g \leqq t \leqq 1} \big(|B_t| - L_t\big)\Big), \\ = \Big(L_g,\ \sup_{g \leqq t \leqq 1} |B_t| - L_g\Big),$$

since $B_g = 0$ and $L_t = L_g$ for $g \leqq t \leqq 1$ by (14).

Thus

$$\mathbb{D}_1 = B_\sigma - \inf_{\sigma \leqq t \leqq 1} B_t = B_\sigma + \sup_{\sigma \leqq t \leqq 1} (-B_t) = B_\sigma + \sup_{\sigma \leqq t \leqq 1} (M_t - B_t - M_t) \\ \overset{\text{law}}{=} L_g + \sup_{g \leqq t \leqq 1} |B_t| - L_g = \sup_{g \leqq t \leqq 1} |B_t|,$$

which proves statement (13).

2. Let us prove relation (16). Let $m = (m_u)_{0 \leqq u \leqq 1}$ be a standard Brownian meander [9, p. 468], [11; 12.3.2], [2]:

$$m_u \equiv \frac{1}{\sqrt{1-g}} |B_{g+u(1-g)}|.$$

In view of (13), we obtain that

$$\mathbb{D}_1 \overset{\text{law}}{=} \sqrt{1-g} \sup_{0 \leqq u \leqq 1} m_u. \tag{23}$$

To find the distribution of $\sqrt{1-g}\,\sup_{0 \leqq u \leqq 1} m_u$, we note first of all that $g \overset{\text{law}}{=} 1-g$; moreover, (by "the first arcsine law" [9, p. 106]) the density

$$f_g(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad 0 < x < 1. \tag{24}$$

It is also known that g and $\sup_{0 \leqq u \leqq 1} m_u$ are independent (see [9, p. 468]).

Furthermore, if $\varepsilon \sim \exp(\frac{1}{2})$, i.e., ε is an exponentially distributed random variable with parameter $\frac{1}{2}$ and does not depend on the Brownian motion B, then one can immediately verify that

$$\varepsilon g \overset{\text{law}}{=} N^2,$$

where $N \sim \mathcal{N}(0,1)$, i.e., N has a standard Gaussian distribution with parameters 0 and 1.

Furthermore, it is known (see [9, p. 484] and [2, pp. 3, 6]), that

$$\mathbf{P}\Big\{|N| \sup_{0 \leqq u \leqq 1} m_u \leqq x\Big\} = \operatorname{th}\frac{x}{2}. \tag{25}$$

Thus,

$$\mathbf{P}\{\sqrt{\varepsilon}\,\mathbb{D}_1 \leqq x\} = \mathbf{P}\Big\{\sqrt{\varepsilon g} \sup_{0 \leqq u \leqq 1} m_u \leqq x\Big\} = \operatorname{th}\frac{x}{2}. \tag{26}$$

Set

$$\eta = \frac{1}{\mathbb{D}_1}, \qquad f_\eta(y) = \frac{d}{dy}\,\mathbf{P}\{\eta \leqq y\}.$$

Since ε and η are independent

$$\mathbf{P}\{\varepsilon \leqq x^2\eta \mid \eta = y\} = \mathbf{P}\{\varepsilon \leqq x^2 y\} = 1 - e^{-x^2y/2}.$$

Therefore, by (26),

$$1 - \int_0^\infty e^{-x^2y/2} f_\eta(y)\,dy = \operatorname{th}\frac{x}{2},$$

or, equivalently,

$$\int_0^\infty e^{-x^2y/2} f_\eta(y)\,dy = 1 - \operatorname{th}\frac{x}{2} = \frac{2e^{-x}}{1+e^{-x}} = 2\sum_{k=0}^\infty (-1)^k e^{-(k+1)\,x}. \tag{27}$$

Let $T_a = \inf\{t \geqq 0\colon \beta_t = a\}$, where $\beta = (\beta_t)_{t\geqq 0}$ is a Brownian motion. It is known (see, for example, [9, p. 68], [10, pp. 302, 303]) that the probability distribution $F_{T_a}(t) = \mathbf{P}\{T_a \leqq t\}$, $a > 0$, has the density

$$f_{T_a}(t) = \frac{a}{\sqrt{2\pi t^3}}\, e^{-a^2/(2t)}, \quad t > 0,$$

and the Laplace transform $\mathbf{E}\, e^{-\lambda T_a} = e^{-a\sqrt{2\lambda}}$, $\lambda > 0$. Therefore, choosing $\lambda = x^2/2$, we find that, for $x \geqq 0$,

$$e^{-ax} = \int_0^\infty \frac{a}{\sqrt{2\pi t^3}}\, e^{-a^2/(2t)} e^{-x^2t/2} dt.$$

From here and (27),

$$\begin{aligned}\int_0^\infty e^{-x^2y/2} f_\eta(y)\,dy &= 2\sum_{k=0}^\infty (-1)^k e^{-(k+1)x} \\ &= 2\int_0^\infty e^{-x^2y/2} \sum_{k=0}^\infty (-1)^k \frac{k+1}{\sqrt{2\pi y^3}}\, e^{-(k+1)^2/(2y)} dy,\end{aligned}$$

and since $x > 0$ is arbitrary, we find a unique solution

$$f_\eta(y) = 2\sum_{k=0}^{\infty}(-1)^k \frac{k+1}{\sqrt{2\pi y^3}}\, e^{-(k+1)^2/(2y)}, \quad y > 0.$$

At last, for $x > 0$

$$F_{\mathbb{D}_1}(x) = \mathbf{P}\{\mathbb{D}_1 \leqq x\} = \mathbf{P}\left\{\eta \geqq \frac{1}{x^2}\right\} = 1 - \mathbf{P}\left\{\eta < \frac{1}{x^2}\right\}$$

and so

$$\begin{aligned} f_{\mathbb{D}_1}(x) &= \frac{2}{x^3}\, f_\eta\left(\frac{1}{x^2}\right) = \frac{4}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(-1)^k(k+1)\, e^{-(k+1)^2x^2/2} \\ &= \sqrt{\frac{8}{\pi}}\sum_{k=1}^{\infty}(-1)^{k-1}k\, e^{-k^2x^2/2}, \end{aligned}$$

which proves relation (16).

3. The relation $\mathbf{E}\,\mathbb{D}_1 = \sqrt{8/\pi}\,\log 2$ can be obtained from (16):

$$\begin{aligned} \mathbf{E}\,\mathbb{D}_1 &= \int_0^\infty x f_{\mathbb{D}_1}(x)\,dx = \sqrt{\frac{8}{\pi}}\int_0^\infty\left(\sum_{k=1}^{\infty}(-1)^{k-1}kx\, e^{-k^2x^2/2}\right)dx \\ &= \sqrt{\frac{8}{\pi}}\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^\infty kx\, e^{-k^2x^2/2}dx = \sqrt{\frac{8}{\pi}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = \sqrt{\frac{8}{\pi}}\,\log 2. \end{aligned}$$

(In this case it is necessary to justify the possibility of changing the order of integration and summation. A similar change is easy if one operates with $\mathbf{E}\,\mathbb{D}_1 I(a < \mathbb{D}_1 < A)$, where $0 < a < A < \infty$, instead of $\mathbf{E}\,\mathbb{D}_1$. Then one should use the fact that $\mathbf{E}\,\mathbb{D}_1 I(a < \mathbb{D}_1 < A) \uparrow \mathbf{E}\,\mathbb{D}_1$ for $a \downarrow 0$, $A \uparrow \infty$.)

We can also obtain the relation $\mathbf{E}\,\mathbb{D}_1 = \sqrt{8/\pi}\,\log 2$ from the following consideration.

From (23)

$$\mathbf{E}\,\mathbb{D}_1 = \mathbf{E}\,\sqrt{1-g}S_m = \mathbf{E}\,\sqrt{1-g}\,\mathbf{E}\,S_m, \tag{28}$$

where $S_m = \sup_{0\leqq u\leqq 1} m_u$ and the second equality follows from the independence property of the values g and S_m already noted above.

By (24)

$$\mathbf{E}\,\sqrt{1-g} = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x}} = \frac{2}{\pi}.$$

In order to find the expectation $\mathbf{E}\,S_m$ in (28), we use the relation (see (25)) $\mathbf{P}\{|N|\,S_m \leqq x\} = \mathrm{th}\,(x/2)$. Then we find that

$$\mathbf{E}\,|N|\,S_m = \int_0^\infty\left(1 - \mathrm{th}\,\frac{x}{2}\right)dx = \int_0^\infty \frac{e^{-x/2}dx}{\mathrm{ch}\,(x/2)} = 4\int_0^1 \frac{y\,dy}{1+y^2} = 2\log 2.$$

Since N and S_m are independent

$$\mathbf{E}\,|N|\,S_m = \mathbf{E}\,|N|\,\mathbf{E}\,S_m = \sqrt{\frac{2}{\pi}}\,\mathbf{E}\,S_m.$$

So

$$\mathbf{E}\,S_m = \sqrt{\frac{\pi}{2}}\,\mathbf{E}\,|N|\,S_m = \sqrt{\frac{\pi}{2}}\,2\log 2 = \sqrt{2\pi}\,\log 2,$$

and, therefore,

$$\mathbf{E}\,\mathbb{D}_1 = \mathbf{E}\,\sqrt{1-g}\,\mathbf{E}\,S_m = \frac{2}{\pi}\,\sqrt{2\pi}\,\log 2 = \sqrt{\frac{8}{\pi}}\,\log 2.$$

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REFERENCES

[1] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion. Facts and Formulae*, Birkhäuser, Basel, 1996.
[2] Ph. Carmona, F. Petit, J. W. Pitman, and M. Yor, *On the laws of homogeneous functionals of the Brownian bridge*, Prepublication No. 441, Paris, Laboratoire de Probabilités de l'Université Paris VI, April 1998.
[3] C. Donati-Martin and Z. Shi, M. Yor, *A* $[0,1]^2$*-valued Brownian functional with arcsine marginals*, Prepublication No. 445, Paris, Laboratoire de Probabilités de l'Université Paris VI, April 1998.
[4] J. L. Doob, *Heuristic approach to the Kolmogorov–Smirnov theorems*, Ann. Math. Statistics, 20 (1949), pp. 393–403.
[5] W. Feller, *On the Kolmogorov–Smirnov limit theorems for empirical distributions*, Ann. Math. Statistics, 19 (1948), pp. 177–189.
[6] W. Feller, *The asymptotic distribution of the range of sums of independent random variables*, Ann. Math. Statistics, 22 (1951), pp. 427–432.
[7] A. Kolmogoroff, *Sulla determinazione empirica di una legge di distribuzione*, Giorn. Instit. Ital. Attuari, 4 (1933), pp. 83–91.
[8] J. W. Pitman and M. Yor, *Quelques identités en loi pour les processus de Bessel*, Astérisque, No. 236, 1996, pp. 249–276.
[9] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin, 1994.
[10] A. N. Shiryaev, *Essentials of Stochastic Finance*, World Scientific, Tokyo, 1999.
[11] M. Yor, *Some Aspects of Brownian Motion. Part* I: *Some Special Functionals. Part* II: *Some Recent Martingale Problems*, Birkhäuser, Basel, 1992, 1997.