Chaos and Bifurcation in 2007-08 Financial Crisis

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Abstract

The impact of increasing leverage in the economy produces hyperreaction of market participants to variations of their revenues. If the income of banks decreases, they mass-reduce their lendings; if corporations sales drop, and due to existing debt they cannot adjust their liquidities by further borrowings, then they must immediately reduce their expenses, lay off staff, and cancel investments. This hyperreaction produces a bifurcation mechanism, and eventually a strong dynamical instability in capital markets, commonly called systemic risk. In this article, we show that this instability can be monitored by measuring the highest eigenvalue of a matrix of elasticities. These elasticities measure the reaction of each sector of the economy to a drop in its revenues from another sector. This highest eigenvalue - also called the spectral radius - of the elasticity matrix, can be used as an early indicator of market instability and potential crisis. Grandmont (1985) and subsequent research showed the possibility that the "invisible hand" of markets become chaotic, opening the door to uncontrolled swings. Our contribution is to provide an actual way of measuring how close to chaos the market is. Estimating elasticities and actually generating the indicators of instability will be the topic of forthcoming research.

Keywords systemic risk, systemic crisis, econophysics, macroeconomics, bifurcation, system stability, chaos

1 Introduction

The global aspect of the subprime-originated financial crisis in 2007-08 is the contagion of risks which is better described by *the butterfly effect*, which started regularly being mentioned after the series of financial "unthinkables" that took place in September of 2008, starting with the nationalization of government sponsored enterprises Freddie Mac and Fannie Mae, the demise of the investment bank Lehman Brothers a week later, the fire-sale of another one Merrill Lynch to Bank of America on the same day, and the government bail-out of the insurance giant AIG just two days later.

"Butterfly effect" is a term used to describe a phenomenon such that small changes at the initial stage result in a huge difference in long-term behavior. The current financial crisis started in the U.S. real estate market and spread to all over the world, and people are still debating when and how this crisis will be over. Such a phenomenon is formally defined in dynamical systems as *sensitive dependence on initial condition*. When a dynamical system possesses a sensitive dependence on initial condition together with cyclical behavior, the system often exhibits *chaos*, de Melo and Palis (1982).

In dynamical systems theory, a *bifurcation* refers to a structural modification of the system behavior upon a continuous change in the parameters of its equations. A *catastrophe* occurs when, following a bifurcation, a small change in parameters discontinuously alters the equilibrium state of the economy. During the 2007-08 crisis, we did observe such a catastrophic event, where a mild evolution of economic parameters ended into a drastic shift in financial interactions. Before the 2007 subprime crisis, the economy was in what physicists call a "meta-stable equilibrium", that is, an equilibrium state that is destroyed by a very small perturbation – like a dry forest totally burning upon the scratch of a match – leading to a series of catastrophic events, until another basin of attraction is reached, i.e. another stationary evolution mode, another cycle or, even, a strange attractor as chaos theory predicts.

In this paper we suggest that the current financial crisis was mainly caused by a breakdown of the dynamic stability of the financial system, according to some catastrophic mechanism. More precisely, we start from a mathematical model in $\mathbb{R}^n$ (the dimension n will be specified in the next section) of the financial system that exhibits a stationary state equilibrium. The financial activities are considered as continuous perturbations of this equilibrium: when the perturbation is small enough, the equilibrium persists and the economy remains stable. When the perturbation is too big, the equilibrium collapses and a financial crisis emerges. Furthermore, we show that the critical size of perturbations that destroy the equilibrium shrinks when financial actors react more rapidly and intensely to other actors they are in business with, leading to a meta-stable equilibrium and a catastrophe. The critical perturbation size is directly related to the debt and borrowing capacity, the leverage, and the market liquidity. In other words our mathematical model shows the causal relation between leverage and market instability.

Based on these observations, we propose the principles of methodology to build an early indicator of the global system instability. The details of such indicator still need to be worked out and tested, as all economic indices involved in this methodology are not readily available.

In Section 2 we provide an intuitive view of the chaos in the current financial crisis and relevant mathematics background. Section 3 will be fully devoted to the structural stability and perturbation analysis of the financial system.

2 Glimpse of Chaos

Financial crisis is generally defined as a situation in which some financial institutions or assets suddenly lose a large part of their value. The current (2007 - 2009+) crisis started in a small sector of global economy called "the U.S. real estate market". In the United States housing bubbles started to form due to low domestic interest rates and the trade deficit which resulted in large foreign capital inflows. These two factors made easy and inexpensive credit available, and many people started investing in real estate. The Case-Shiller Home Price Index had its peak in the second quarter of 2006 [14], and the U.S. house prices have steadily decreased since. However it is not only the U.S. houses that lost value. Many banks and financial entities, both regional and global, went bankrupt. Many companies, both small and large, went under as well. In both cases the main cause was solvency and liquidity. During the development of the crisis, the damage seemed to be only getting more severe, massive, and unpredictable. Third year in the crisis, the current situation seems to be stabilizing, but the future still looks unpredictable. In the meantime, the blame has been aimed at financial engineers (also known as "quants") for having created esoteric financial derivatives and used faulty mathematical models to evaluate them.

Avoiding the question of model validity, which appears to us as a side question, we try to understand the financial, then economic crisis in its dynamical aspects.

Here we get a glimpse of *chaos*: what started locally has spread globally with unpredictable severity, and this suggests *sensitive dependence on initial condition*; the mathematical models which used to work well in the past do not work all of a sudden, and this hints *bifurcation* of the system.

Typically chaos is found in dynamical systems that possess nontrivial recurrence (i.e. which cannot be isolated), and indeed there is "recurrence of risks" behind the current financial crisis. In dynamical systems theory recurrence is produced by the feedback loop, which in finance became global due to securitization. Although the original purpose of the securitization was to diversify default risk, this "originate to distribute" practice spawned too many risky loans which were destined to default. As a result the risk was disseminated globally as opposed to diversified, then boomeranged back to the issuer of the loans as well as to the borrowers. This is because all the financial transactions were made in a closed system. If the Masters of the Universe on our planet had managed to pass all those risks to other denizens of the universe, we would not be having any of the problems we are having now, for the Earth aggregately acts as a *source* of risks. This was for instance the case of local systemic crisis such as the Asian-Russian one in 1997-98, which eventually was rapidly absorbed.

To visualize the situation, let us consider the following feedback model. If we consider the so-called fixed and variable cash flows among financial segments during the real estate boom, cash flows that one usually calls "fixed" are the scheduled ones, such as salaries, contributions to (pension and other investment) plans, coupons, installments, etc. Other cash flows are said "variable" because they are at will: investments, loans, dividends, savings, etc. In fact, both types of cash flows are impacted by economic conditions, the less variable ones not necessarily being those called "fixed". For example, the cash flow from households to industry in exchange of goods and services remains almost constant regardless of the economic condition. Noninterest income for banks such as ATM fees and credit card fees are not affected by economic deterioration, for banks can always raise those fees. Such cash flows are rather steady, while payments from subprime loans, which are subject to default, are in practice more variable cash flows.

At a macroeconomic level, variations of aggregate cash flows could be explained by shifts in the classical Hick's IS-LM curves. However in order to assess the actual market stability, we do not deduce variations of cash flows from a model, but from empirical observations. Market instability, which is our core target of study, may possibly result from a behavior that is predicted by a model, e.g. Grandmont (1985), but it may also well be the consequence of the economy departing from classical models.

Let us now broadly divide the economy into several segments in the spirit of [5]. Typically home buyers (HB), which we do not distinguish from general consumers, get financing from local mortgage lenders (ML) which include the mortgage divisions of large banks. To facilitate financing with limited fund, these mortgages are sold to large banks (LB - not only banks but other financial institution that function like banks, for example brokers and insurance companies) and the government sponsored enterprises (GSE - Fannie Mae and Freddie Mac) for securitization - they are sliced, diced, and repackaged as mortgage backed securities (MBSs) which is a special kind of asset backed securities (ABSs) or collateralized debt obligations (CDOs). Part of these MBSs are kept within the banks (super-senior tranches) or are securitized again (ABS-squared) and sold in secondary mortgage market. They are also sold to investors. The investors (I) consist of many funds from all over the world, such as pension funds, mutual funds, academic endowments, state employees' retirement funds, sovereign funds etc. Many people invest, directly and indirectly, in such funds, so the impact of these funds on the real sector of economy is immense. Speculative hedge funds are here excluded, since in this framework they are just intermediaries, hence their presence merely affects the whole cycle. Alternatively, they can be considered as part of the "I" class. Another key player in the real sector is corporations (C). Home buyers pay their mortgage installments to mortgage lenders and large banks, and they do so largely thanks to the wages paid mostly by corporations which in return, get financed mostly by large banks and investors.

We also have here excluded the government as a financial actor, as we wish to understand the dynamics of the free market. Government actions, when they occur, are directed mainly to banks and government sponsored enterprises, and indirectly to corporations and consumers by the fiscal policy. The framework we describe here provides a clear basis to understand the potential impact of such or such government actions.

The cash flows among these six segments, HB, ML, LB, GSE, C, and I, can be further classified into two groups, variable cash flows and fixed cash flows as explained previously.

Variable cash flows include equity investments, debt investments (commonly called loans), and dividends which include payments that act like dividends.

1. Equity investment

   (i) HB to HB: home buyers invest in houses, and sell those to one another.

   (ii) C to C: companies invest into each other.

   (iii) I to LB, GSE, C: investors buy stocks of LB, GSE, and C.

2. Loan (debt investment)

   (i) ML to HB: mortgage loans

   (ii) LB to HB: credit cards and other financing
   LB to ML: purchasing mortgages for securitization
   LB to LB and GSE: secondary MBS market
   LB to C: bank loans to companies

   (iii) GSE to ML, LB: guarantees mortgages by purchasing them and creating MBS.

   (iv) I to LB, GSE, C: investors buy bonds issued by LB, GSE, and C.

3. Dividends

   (i) LB, GSE, C to I: investors earn dividends from the LB, GSE, and C stocks they invested.

   (ii) I to HB: investors pay pensions to their client, which work like dividends.

Fixed cash flows are coupons, which include not only bond coupons but payments from fixed rate mortgages and other conventional loans, minimum payments for adjustable rate mortgages and credit card loans, salaries, and contribution to retirement fund and other money market funds. Also included are premiums for credit default swaps (CDSs) to issuing financial institutions from counterparties.

1. Coupons

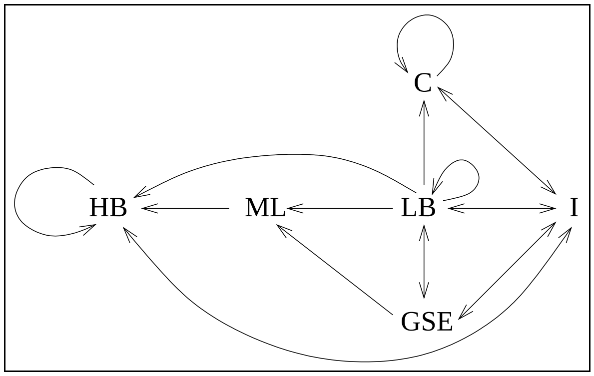

Figure 1: Variable cash flows

(i) HB to ML, LB: mortgage and other financing payments, credit card debt payments

(ii) ML to LB, GSE: although LB and GSE purchase mortgages from ML for securitization and guarantee them by holding the resulting MBSs, the payments from HB are still directly made to the original lender ML. So we interpret that there is a cash flow from ML to GSE and LB.

(iii) LB to LB, GSE, I: coupons for MBS and ABS of MBS markets. Also included are CDS premiums.

(iv) GSE to LB, I: GSE pays coupons to MBS investors. When GSE started buying and guaranteeing more MBSs the financial crisis had already developed [8]. We consider cash flows during normal economy, so do not include this special case which will create a loop from GSE to itself.

(v) C to LB, I: companies pay coupons to the bond holders.

2. Salary: C to HB

3. Contributions: HB to I

When the real estate bubble burst in 2007, home buyers started getting behind their payments. So the financial segments at the receiving end of cash flows involving MBS experienced significant default and write downs (Figure 2), which lead to the bail out of Fannie Mae and Freddie Mac in September of 2008. The blow was more severe for large banks, for they not only invested in those CDOs but also insured against them by selling CDSs. Just a week later Lehman Brothers declared bankruptcy and Merrill Lynch was sold to Bank of America on the same day. Several days later, the insurance giant AIG was bailed out.

The September 20008 saga froze market liquidity, and corporations started not being able to borrow money. This resulted in mass bankruptcy of companies of all size, and consecutively massive unemployment. The market value of

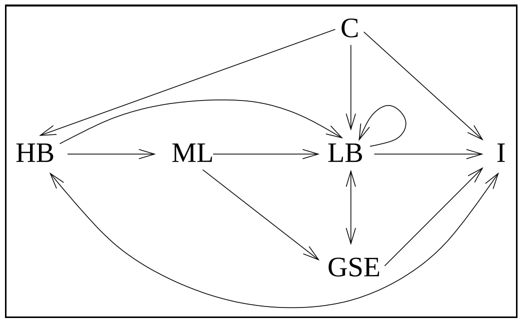

Figure 2: Fixed cash flows

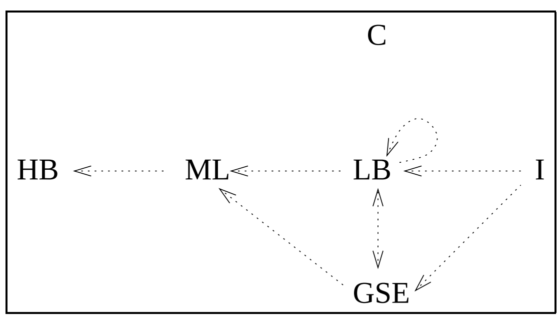

Figure 3: MBS-related funding cash flows

companies and banks plunged, and so did the return to the investors and the home buyers who contributed to them. This resulted in "victims" of the financial crisis. In the U.S. real estate market, not all home buyers contributed to the crisis. There are people who used to be prime lenders and never delinquent in their payments before, but ended up falling behind due to the side effects of the financial crisis, such as plunging house price, soaring interest rates on loans, and job loss [6] [13]. Commercial mortgage holders are like prime lenders in that sense. Likewise there are banks which used to serve prime lenders and people with sound financial basis but eventually fell victim to the crisis as well. Although it is tempting to distinguish the "sinners" and "victims" of the financial crisis, in the end they are not distinguishable at all. Dynamically, once the crisis is on, whether a default comes from an initially insolvent borrower or from a borrower who became insolvent doesn't matter in the evolution of the crisis. The purpose of this paper is to think of not the responsibilities but the evolution of the financial crisis and the transition from equilibrium to chaos.

Thus we have the following feedback loop. The arrows represent the cash flows which experience significant drop (this would be our definition of "default"). This is the least bad scenario, that is, their counter cash flows stay normal. The worst scenario would be that all cash flows be in default. Such a case would have occurred with high probability without government intervention.

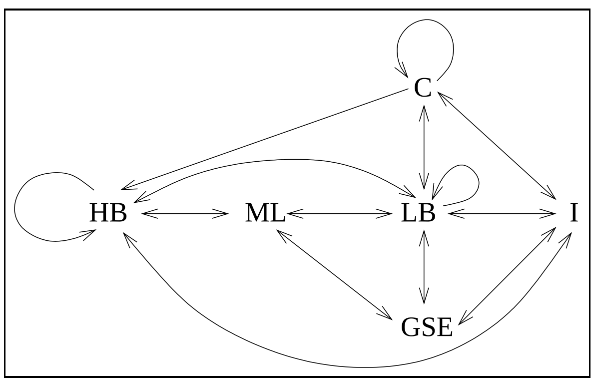

Figure 4: Combined cash flows and related default risk

Assign numbers 1 to 6 to HB, ML, LB, GSE, C, and I in that order, and define a transition matrix $C = (c_{ij})_{1 \leq i,j \leq 6}$ such that

$$c_{ij} = \begin{cases} 1 & \text{if there is cash flow risk going from } i \text{ to } j \\ 0 & \text{if there is no cash flow risk going from } i \text{ to } j \end{cases}$$

By cash flow risk we mean the variability of cash flows from one segment to another due to variations in the income of paying segment.

Using the cash flow chart, we get

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad (1)$$

One could consider that, in average, virtually every cash flow in the picture depends on economic conditions and is impacted by variations of any of the other cash flows. However, when a shock is applied, the particularity of "cash flow risks" as we define it is that cash flows experience brutal changes of correlations and of sensitivity to one another, as soon as some thresholds are being triggered. In this context, as we shall see later in the article, we can consider that the matrix C represents the acceleration of sensitivities upon a shock.

All the entries of the matrix C become positive after two iterations, $C^2 > 0$. This means that starting from one segment, all other segments can be eventually reached. This property implies two things when applied to financial crisis. First, the damage from bad loans will return to the issuers of those loans, as well as the borrowers, and due to the nonlinearity of the system, the degree of damage will be far more severe than that of the initial benefit (fees and commissions). Second, default which took place in any one sector will eventually affect all other sectors through domino effect. This explains why damage in the financial sector spread to the real sector and how defaults in the U.S. real estate market resulted in a global financial crisis.

Compare the 2007-2008 financial crisis with the Asian-Russian crisis from 1997-98. The crisis started in Thailand with the financial collapse of Thai baht when, after severe speculative attacks in May 1997, the Thai government gave up protecting its currency and decided to float the exchange rate. This resulted in a rapid depreciation of Thai baht and an exodus of foreign capital. This created turmoil in the currency market, and the effort to keep the currency from declining further led to domestic interest rate hike. Consequently, the Thai stock market dropped 75% and many companies bankrupted. Other Asian countries such as Indonesia, South Korea, and the Philippines followed its path [15]. The subsequent drop in commodity prices eventually induced Russia into default

Although the series of national defaults in Asia, then in Russia caused panic and hit stock markets worldwide, the dynamic structure of the events is quite simple. Those Asian countries collectively act as an isolated point which interacts with investors (I) via equity investment, with large banks (LB) via loans, and with Russia via commodity trading. (Figure 5). There was no financial instrument backed by assets of the stricken countries and resulting feedback loop. Although there was a brief worldwide stock market crash [16], there was not the kind of chaos observed in the 2007-08 crisis.

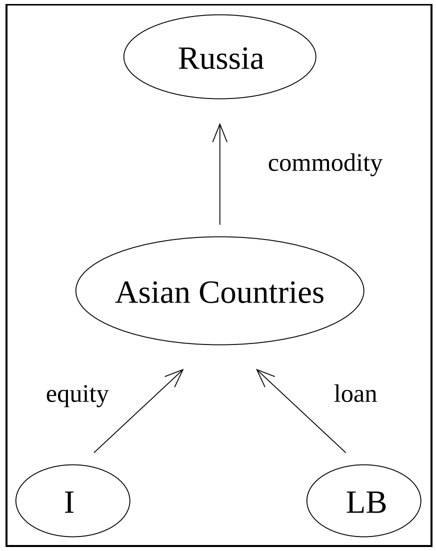

Figure 5: Cash flows among stricken countries and foreign investors

At variance, the March 2000 Nasdaq collapse and the burst of the tech bubble could be considered as seed of the current financial crisis. We could summarize the mechanism as follows.

(1) Venture capitalists (VCs) and private equity investors, followed by general investors incurred huge losses due to coarse mispricing of internet securities.

(2) In order to catch back, they had to show rapid profits and, for this purpose, started investing in a leveraged manner. Typically, VCs invested in LBO (leveraged buy out) structures rather than straight equity, pushing SMEs (small medium enterprises) to massively borrow and leverage on their existing balance sheet.

(3) At the same time, the banking industry developed the CDS and CDO markets, hence easing access to credit by corporations in general, making all these LBO structures possible and profitable in the short term.

Obviously, this put the world at extreme sensitivity to the slightest increase of credit spreads, as a vast portion of the economy suddenly became non profitable because of the debt burden.

3 Stability of Financial Equilibrium

3.1 Propagation of Wealth

Keeping the mathematical intuition from the previous section in mind, we start a formal stability analysis of a financial equilibrium. We model the economy in $\mathbb{R}^n$. The dimension n is the number of main segments of the financial system under consideration, and in our paper it is assumed to be 6 in accordance with the feedback loop in Figure 2, although we keep the letter n for generalization purpose.

Let the wealth vector $w(t)$ represent the wealth of all segments at time t, $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))$ where $w_i(t)$ is the wealth of the market segment i, and let f be a dynamical system of wealth in $\mathbb{R}^n$ such that $f(w(t)) = w(t+1)$. The global wealth $S(w)$ is the sum of all wealth, thus at a given time t,

$$S(w(t)) = \sum_{i=1}^{n} w_i(t) \tag{2}$$

In normal equilibrium condition, the global wealth has a mild growth. We will show that when the economy is too much leveraged, the chance for the global wealth to experience strong downfalls is high due to a bifurcation in the dynamical system.

From time t to $t+1$, let $f_{ij} = f_{ij}(w)$ be the percentage of wealth transferred from the market segment j to i, that is, the amount of wealth transferred from the segment j to i at t is $f_{ij}(w(t))w_j(t)$. Under normal circumstances the economy has a natural internal growth. Let $\gamma_j = \gamma_j(w)$ be the internal growth factor of the segment j. The wealth within each segment $w_j(t+1)$ can be computed from the in and out flows, and the internal growth is as follows.

$$w_i(t+1) = (1+\gamma_i)\left(w_i(t) + \sum_{j=1}^{n} f_{ij}w_j(t) - \sum_{k=1}^{n} f_{ki}w_i(t)\right)$$

Using these notations we can express the wealth function f during the time interval $[t, t+1]$ as

$$w(t+1) = f(w(t)) = (I+\Gamma)(I+F-\overline{F})w(t)$$

where I is the $n \times n$ identity matrix, $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a diagonal matrix with entries $\gamma_i, 1 \le i \le n$,

$$F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & & \ddots & \\ f_{n1} & & \cdots & f_{nn} \end{pmatrix}$$

and $\overline{F}$ is a diagonal matrix whose entries are $\sum_{k=1}^{n} f_{ki}$ for $1 \leq i \leq n$,

$$\overline{F} = \mathrm{diag}\left(\sum_{k=1}^{n} f_{k1}, \sum_{k=1}^{n} f_{k2}, \cdots, \sum_{k=1}^{n} f_{ki}\right)$$

Note that this dynamical system is still nonlinear as coefficients f_{ij} and γ_i depend on the wealths w_i's.

3.2 Perturbation Analysis

We start from the assumption that there is an equilibrium situation characterized by a fixed point $\overline{w}$ of f modulo the internal growth, that is, a stationary state $\overline{w}$ such that

$$f(\overline{w}) = (I + \Gamma)\overline{w}$$

which means that at this equilibrium the income equals the expenditure,

$$(F - \overline{F})(\overline{w}) = 0$$

This $\overline{w}$ can be seen as a fixed point of $\overline{f} = (I + \Gamma)^{-1} f$. We shall make the assumption that in the initial state of the economy, $\overline{w}$ is a stable fixed point. In other words small perturbations are absorbed and the economy converges back to the equilibrium.

The global wealth growth equation Equation 2 means that

$$S(\overline{f}(\overline{w})) = S(\overline{w}) \tag{3}$$

implies that one of the eigenvalues of $T = I + F - \overline{F}$ is 1. The stability of $\overline{w}$ means that all other eigenvalues have modulus less than 1.

In real life, a stationary equilibrium is always perturbed, so unexpected changes of wealth occur frequently. Here we mean by unexpected change a digression from the equilibrium, which is caused by changes of net cash inflows and equity levels.

A drop in cash inflow can occur in so-called variable payments upon decision of financial actors, but it can as well occur in so-called fixed payments, for instance due to the default, job loss etc.

Let $x_j(t)$ be the drop in wealth of the segment j at time t and $a_{ij} = a_{ij}(t)$ be the elasticity coefficient such that if j experiences a decreased wealth at time t, then the cash flow from j to another segment i is reduced by $a_{ij}x_j$. Elasticities depend among others on borrowing capacities, which again depend on two things, leverage and credit rating. Then the wealth w_i near the equilibrium $\overline{w}$

has the following dynamics, in which the internal growth is not considered:

$$w_i(t+1) = w_i(t) + \left(\sum_{j=1}^{n} f_{ij} w_j(t) - \sum_{k=1}^{n} f_{ki} w_i(t) \right) \quad (4)$$

$$- \left(\sum_{j=1}^{n} a_{ij} x_j - \sum_{k=1}^{n} a_{ki} x_i \right) \quad (5)$$

This equation, however, reflects only the drop in cash flow due to changes in counterparty payments and not the internal drop of equity level. For example, an overnight drop of bank stock prices due to fear for bank runs reduces the aggregate wealth of banks, but this is not due to reduced cash inflows of banks. To accommodate this internal drop of wealth, we introduce an equity drop factor $\delta_i = \delta_i(t)$. The amount $\delta_i x_i(t)$ measures the reduction in wealth of the segment i at time t, which will consequently lead i to reduce its payments at time $t+1$. Then the change of wealth $x_i(t+1)$ of i at time $t+1$ can be expressed as the sum of internal equity drop and external counterparty default,

$$x_i(t+1) = \delta_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) - \sum_{k=1}^{n} a_{ki} x_i(t) \quad (6)$$

and

$$w_i(t+1) = w_i(t) + \left(\sum_{j=1}^{n} f_{ij} w_j(t) - \sum_{k=1}^{n} f_{ki} w_i(t) \right) - x_i(t+1) \quad (7)$$

Note that this drop in equity is itself the result of the dynamics among asset managers and traders when confidence disappears in the stocks of a given sector. In this paper, we will not model this dynamics and simply consider such an event as shock in the market, for we are more interested in the result of such shock.

Define Δ to be a diagonal matrix whose entries are δ_i's,

$$\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_i, \ldots, \delta_n)$$

Let A be an $n \times n$ matrix with entries a_{ij}, $A = (a_{ij})_{1 \le i,j \le n}$, and let $\overline{A}$ is a diagonal matrix whose entries are $\sum_{k=1}^{n} a_{ki}$ for $1 \le i \le n$,

$$\overline{A} = \mathrm{diag}\Big(\sum_{k=1}^{n} a_{k1}, \ldots, \sum_{k=1}^{n} a_{ki}, \ldots, \sum_{k=1}^{n} a_{kn}\Big)$$

Let $B = \Delta + A - \overline{A}$, then by Equations (6) and (7)

$$x(t+1) = Bx(t) \quad (8)$$

and

$$w(t+1) = Tw(t) - Bx(t) \tag{9}$$

As long as x is infinitesimal, this matrix B is the Jacobian matrix of partial derivatives and x_j is the differential dw_j of the wealth w_j. More precisely, $\overline{f}(w) = T(w)w$ implies

$$Df(w)x = T(w)x + Df(w)xw$$

So the Jacobian matrix B is

$$B = T + DT^{\top} w$$

Let $T = [t_{ij}]_{1 \le i,j \le n}$, then the instantaneous rate of change b_{ij} of the discounted cash flow from the segment j to i,

$$b_{ij} = \frac{\partial \overline{f}(w(t))_i}{\partial w_j(t)} = \frac{\partial w_i(t+1)}{\partial w_j(t)} = t_{ij} + \frac{\partial t_{ij}}{\partial w_j} w_j$$

The matrix B may take a different shape if it represents a reaction of the various market segments to a sudden shock in inflows. The equilibrium $\overline{w}$ is a fixed point of the function $\overline{f}$, thus an eigenvector of T associated with eigenvalue 1. Initially, we assume this equilibrium to be stable, which implies that the eigenvalues of $B(\overline{w})$ have modulus smaller than 1. When leverage increases and/or the global wealth of the sectors decreases, the borrowing capacity drops immediately, and the elasticities tend to increase sharply. This concavity is an effect of the overreaction of market participants under liquidity shortage.

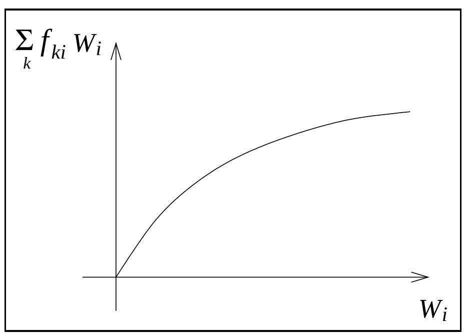

Figure 6: The graph of outflow vs. wealth for segment i

When the market is highly leveraged, the elasticities reach such a level that one or more of the eigenvalues of B have modulus above 1.

In this case, perturbations propagate. For instance, if variations in the money flow are due to some default in payments, then default becomes structurally installed. This is the situation we now have: government's bailouts of large banks and corporations or attempts to restructure home mortgages are evidences of installed defaults.

4 Financial Crisis: Breakage of Stability

In this section we investigate how the result of Section 3.2 can be applied to the current financial crisis which we consider having started in 2007.

More precisely, if one of the segment i drops in value, then what would be the impact of this event on the other segments at the next time step? In the spirit of Section 3.2, we can have two possibilities. First, the equilibrium is an attractor, so eventually the perturbation is absorbed. Second, the perturbation breaks the stability and propagates through all the segment.

A remarkable fact is that the entries of the perturbation matrix DB are directly related to leverage. When the segments are deep in debt, the elasticities are high, so B has larger entries, therefore has a bigger chance of having at least one eigenvalue greater than 1. We conclude that when leverage is high, instability increases and market equilibrium becomes unstable.

Once we enter an unstable stage, what happens next? When there is a bifurcation that breaks stability, different possibilities may occur:

(1) The equilibrium becomes hyperbolic with at least one of the eigenvalues becoming a real number greater than 1. In this case, the market shifts towards another attracting equilibrium or a more complicated attractor.

(2) The largest eigenvalues are a pair of complex conjugate numbers with modulus greater than 1. This is called an "Andronov-Hopf bifurcation", in which case the equilibrium becomes a cycle.

In general, after a crisis, even if the market would a priori follow the catastrophic path towards a new, deterred, equilibrium, such as deflation for instance, it is very probable that the government policy will consist in doing everything in its power to avoid such a shift to lock-in, therefore putting the economy into a cyclic behavior. As a consequence, in either case, one can expect that, posterior to a crisis, the economy enters a period of intense oscillations.

Normally, the frequency of oscillations should be related to the imaginary part of the eigenvalues. However, because of year end tax reporting and usually yearly investment planning, the market tends to be subject to a forced, rather than free, oscillator, with yearly frequency.

More precisely, in the case of Andronov-Hopf bifurcation, there is an invariant closed curve (cycle) near the original equilibrium. Although the wealth transition matrix $T(t)$ and the perturbed one $(T_D B)(t)$ are linear during the time period $[t, t+1]$, the wealth propagation over all time is nonlinear. Due to this nonlinearity, the system will cycle for a while, resulting in lots of bouncing back. As the financial crisis develops, the wealth of some of the segments is low and the linear approximation does not work. Economic recovery will go through a cycle, and each of the market segment will behave in an oscillating manner.

We will see a sequence of growth and recession of each segment until the system stabilizes and reach a new equilibrium of wealth $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_n)$.

In the other case of an equilibrium shift, the government action is the major source of nonlinearity, and the above scenario occurs in a similar manner.

5 Numeric Examples

In this section we provide numeric examples that utilizes the analysis in Section 3.2. The examples are purely hypothetical, yet they give a glimpse of various possibilities in real life situation. We follow the notations in Section 2 and assume there are six market segments, home buyers (HB), local mortgage lenders (ML), large banks (LB), corporations (C), and investors (I) and consider combined cash flows (Figure 2) among them. We study two cases: one with high leverage in every sector for every cash flow, the other which mimics the evolution of the 2007-08 crisis.

In each example, we specify values of the elasticities a_{ij}'s, then build B, and compute its highest eigenvalue. Although the elasticities are hypothetical, we can see that they do not need to reach very high levels for the matrix B to have the highest eigenvalue above 1, that is, for the system to be unstable.

Example 5.1. *High leverage in Every Cash Flow.*

During a stable economy, all eigenvalues of B have modulus less than 1 at the equilibrium. For a wealth transition matrix to have eigenvalues between -1 and 1, its entries should be reasonably small after normalization of data. Not only our examples are hypothetical, the main focus of our result is the interaction among the six financial market segments and the resulting chaos. We start with the transition matrix C of the combined cash flow Equation 1. By our construction in Section 3.1, the wealth transition matrix F has the same entries as the transpose of C. Further we set each entry to 0.2 in order to make the eigenvalues between -1 and 1,

$$F = 0.2\, C^{\top}$$

The matrix T for the wealth transition function $\overline{f}$ Equation 3 is therefore

$$T = \begin{pmatrix} -0.4 & 0.2 & 0.2 & 0 & 0.2 & 0.2 \\ 0.2 & -0.4 & 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0.2 & -0.8 & 0.2 & 0.2 & 0.2 \\ 0 & 0.2 & 0.2 & -0.4 & 0 & 0.2 \\ 0 & 0 & 0.2 & 0 & -0.4 & 0.2 \\ 0.2 & 0 & 0.2 & 0.2 & 0.2 & -0.6 \end{pmatrix} \tag{10}$$

This matrix T has an eigenvalue whose modulus is 1 and the rest less than 1, so it represents a stable state. We now assume that all six market segments are highly leveraged. In this case the spectral radius of the Jacobian matrix

B is close to that of T. In order to ensure a stable starting point, we slightly decrease the magnitude of each entry of T. We begin by assuming $B = T$, then modify the internal drop factor Δ.

$$T = \begin{pmatrix} -0.38 & 0.19 & 0.19 & 0 & 0.19 & 0.19 \\ 0.19 & -0.38 & 0.19 & 0.19 & 0 & 0 \\ 0.19 & 0.19 & -0.76 & 0.19 & 0.19 & 0.19 \\ 0 & 0.19 & 0.19 & -0.38 & 0 & 0.19 \\ 0 & 0 & 0.19 & 0 & -0.38 & 0.19 \\ 0.19 & 0 & 0.19 & 0.19 & 0.19 & -0.57 \end{pmatrix} \tag{11}$$

This matrix B has eigenvalues with maximum modulus 0.95, thus the dynamical system is dominated by the wealth transition matrix T. As financial bubbles form and eventually burst, the equity level of each segment changes, which subsequently changes the Jacobian matrix B. Below is a summary. The diagonal matrix added to B is the internal equity drop matrix Δ.

Stage 1 $B = \text{diag}(0, 0, 0, 0, 0, 0) + T$, $\max|\lambda| = 0.95$
Initial equilibrium

Stage 2 $B = \text{diag}(0.25, 0.25, 0, 0, 0, 0) + T$, $\max|\lambda| = 0.9396$
Beginning of subprime crisis: HB, ML equity drop

Stage 3 $B = \text{diag}(0.5, 0.5, 0.25, 0.25, 0, 0) + T$, $\max|\lambda| = 0.7641$
Subprime crisis cont'd: HB, ML equity drop further, LB, GSE report losses, marking down of debts stabilizes market

Stage 4 $B = \text{diag}(1, 1, 0.5, 0.5, 0, 0) + T$, $\max|\lambda| = 0.9573$
Crisis evolution: HB, ML default, LB, GSE report more losses

Stage 5 $B = \text{diag}(1, 0.25, 1.25, 0.75, 0.5, 0.25) + T$, $\max|\lambda| = 1.0104$
Activation of guarantees: ML supported by GSE (March 2008), market destabilizes again

Stage 6 $B = \text{diag}(1, 1, 1.75, 1.5, 0.75, 0.25) + T$, $\max|\lambda| = 1.4687$
Crisis continues (April - August 2008), market is very unstable, forthcoming crisis is inevitable

Stage 7 $B = \text{diag}(1, 0.25, 1.25, 0.75, 0.75, 0.25) + T$, $\max|\lambda| = 1.0286$
Government bails out the "too-big-to-fail"(September 2008). Prevention of systemic failure introduces strong nonlinearity in dynamical systems

Stage 8 $B = \text{diag}(1.25, 0.25, 0.5, 0.75, 0.75, 0.25) + T$, $\max|\lambda| = 1.0360$
Quantitative easing: LB rebound, HB still drop

Stage 9 $B = \text{diag}(1.5, 0.5, 0, 0.75, 1, 0.25) + T$, $\max|\lambda| = 1.2398$
LB got better, C got worse due to lack of credit (January - March 2009), market enters another unstable period

Stage 10 $B = \text{diag}(1.5, 0.5, -0.5, 0.25, 1, 0) + T$, $\max|\lambda| = 1.3603$
Apparent recovery: LB gain wealth, GSE, I get better, instability increases

Stage 11 $B = \text{diag}(1, 0.25, -1, 0, 0.5, -0.25) + T$, $\max|\lambda| = 1.8492$
Overconfidence: Everybody got better, LB, I report positive earning (now), instability at its highest point.

Example 5.2. *High leverage in Selected Cash Flows.*

This example is more realistic than the previous one. Even when one experiences a sharp drop of cash inflow, some cash outflows may remain unaffected. This is due to the difference in payment obligation amount. For example, when one's monthly income drops by \$1,000 due to wage cut, it may affect his mortgage payment of \$2000, but not his IRA contribution of \$150 or monthly bank charges less than \$50 in total. For this reason we set elasticities of cash flows directly affected by the crisis to high values, and the other ones to relatively small values. Let the elasticity matrix A be the following,

$$A = \begin{pmatrix} 1 & 0.01 & 0.01 & 0 & 0.5 & 0.5 \\ 0.5 & 1 & 0.01 & 0.01 & 0 & 0 \\ 0.01 & 0.25 & 1 & 0.25 & 0.25 & 0.01 \\ 0 & 0.01 & 0.25 & 1 & 0 & 0.01 \\ 0 & 0 & 0.5 & 0 & 1 & 0.5 \\ 0.01 & 0 & 0.5 & 0.5 & 0.01 & 1 \end{pmatrix} \tag{12}$$

Due to the cancellation in $A - \overline{A}$ the diagonal elements of A do not affect the overall dynamics of the system, so we assign 1 to each of them. Then we have the following $A - \overline{A}$,

$$A - \overline{A} = \begin{pmatrix} -0.52 & 0.01 & 0.01 & 0 & 0.5 & 0.5 \\ 0.5 & -0.27 & 0.01 & 0.01 & 0 & 0 \\ 0.01 & 0.25 & -1.27 & 0.25 & 0.25 & 0.01 \\ 0 & 0.01 & 0.25 & -0.76 & 0 & 0.01 \\ 0 & 0 & 0.5 & 0 & -0.76 & 0.5 \\ 0.01 & 0 & 0.5 & 0.5 & 0.01 & -1.02 \end{pmatrix} \tag{13}$$

The Jacobian matrix is the sum of $A - \overline{A}$ and the wealth drop matrix Δ. We keep the same Δ as in Example 1 for comparison purpose.

Stage 1 $B = \text{diag}(0, 0, 0, 0, 0, 0) + A - \overline{A}$, $\max|\lambda| = 1.4555$
Market already unstable from the beginning due to high elasticities.

Stage 2 $B = \text{diag}(0.25, 0.25, 0, 0, 0, 0) + A - \overline{A}$, $\max|\lambda| = 1.4397$
Crisis begins, HB, ML equity drop, the maximum eigenvalue slightly decreases

Stage 3 $B = \text{diag}(0.5, 0.5, 0.25, 0.25, 0, 0) + A - \overline{A}$, $\max|\lambda| = 1.2108$
Crisis evolves, HB, ML equity drop further, LB, GSE lose, the maximum eigenvalue decreases further due to marking down of debts

Stage 4 $B = \text{diag}(1, 1, 0.5, 0.5, 0, 0) + A - \overline{A}$, $\max|\lambda| = 1.1554$HB
Subprime loans default, ML default, LB, GSE lose more, maximum eigenvalue decreases

Stage 5 $B = \text{diag}(1, 0.25, 1.25, 0.75, 0.5, 0.25) + A - \overline{A}$, $\max|\lambda| = 0.7305$
Activation of guarantees, ML supported by GSE (March 2008), system seems stabilizing

Stage 6 $B = \text{diag}(1, 1, 1.75, 1.5, 0.75, 0.25) + A - \overline{A}$, $\max|\lambda| = 1.1224$
Crisis continues (April - August 2008), system unstable again

Stage 7 $B = \text{diag}(1, 0.25, 1.25, 0.75, 0.75, 0.25) + A - \overline{A}$, $\max|\lambda| = |-0.6514 + 0.0855i| = 0.6570$
Government bails out (September - October 2008), market a priori would stabilize at recession level, in practice quantitative easing introduced nonlinearities which induced strong oscillations

Stage 8 $B = \text{diag}(1.25, 0.25, 0.5, 0.75, 0.75, 0.25) + A - \overline{A}$, $\max|\lambda| = |-0.8929 + 0.1491i| = 0.9053$
HB drops LB rebounds thanks to quantitative easing, recession continues but stability decreases. Complex eigenvalues suggest cyclic behavior

Stage 9 $B = \text{diag}(1.5, 0.5, 0, 0.75, 1, 0.25) + A - \overline{A}$, $\max|\lambda| = 1.3582$
LB are better off, but HB, ML, C are worsening, stability is broken, but for better or worse?

Stage 10 $B = \text{diag}(1.5, 0.5, -0.25, 0.75, 1, 0.25) + A - \overline{A}$, $\max|\lambda| = 1.6050$
LB recover, the rest stays the same, instability keeps increasing

Stage 11 $B = \text{diag}(1.5, 0.5, -0.5, 0.25, 1, 0) + A - \overline{A}$, $\max|\lambda| = 1.8570$
Markets rally, LB gain more wealth, GSE, I got better, market becoming very unstable keep

Stage 12 $B = \text{diag}(1, 0.25, -1, 0, 0.5, -0.25) + A - \overline{A}$, $\max|\lambda| = 2.3538$
Market rally cont'd, everybody got better, LB, I report positive earning, instability at its highest point

6 Conclusion

In normal market conditions, the risk is usually monitored using techniques such as VaR (value at risk) or the volatility measures, which in some sense measure "the size of the waves" in order to guarantee that a given financial institution can face it. When a crisis occurs, it appears more important to estimate the "distance to the chute" than the "size of the waves", indeed, the dynamic part dominates the random part of the evolution laws. In this article, we addressed this question by trying to identify when the market equilibrium becomes unstable. For this purpose, following classical chaos theory, we look at the so-called Jacobian matrix of the dynamical system near the equilibrium and ask the question of its highest eigenvalue.

The entries of this matrix corresponds to how a given segment of the economy (banks, corporations, investors, consumers etc.) reacts in its spendings and investments to variations of its income. The sensitivity coefficients of the outflows with respect to the inflows, which we call here "elasticities", strongly depend on the borrowing capacities of the financial actors, and their general leverage. When debt-to-wealth ratio is high, these elasticities tend to increase sharply. As a consequence, the highest eigenvalue of the Jacobian matrix passes the instability threshold, putting the market in high risk of turmoil.

We provided two examples of scenarios of transitions from stability to instability as possible explanations of the 2007-08 crisis. In these examples, it is striking that after a period of instability followed by an actual drop in wealth, the market temporarily stabilizes in a recession state. Then actions to exit the recession, such as quantitative easing, put again the market in an unstable state.

If we strictly follow the conclusions of this study, first, one should expect several periods of significant market oscillations: rallies followed by more or less rapid falls. Second, incentive actions such as quantitative easing should be used carefully in view of their long-term effects in order not to be lured by a seeming recovery which is just the upward side of the oscillation.

References

[1] Benhabib, J., 1992. *Cycles and Chaos in Economic Equilibrium*, Princeton University Press.

[2] Benhabib, J. and Day, R. H., 1982. A characterization of Erratic dynamics in the overlapping generations model. *Journal of Economics Dynamics and Control*, 4, 37-55.

[3] Brin, M. and Stuck, G., 2002. *Introduction to Dynamical Systems*, Cambridge University Press.

[4] Benhabib, J. and Nishimura K., 1979. The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal Economic growth. *Journal of Economic Theory*, 21, 421-444.

[5] Benhabib, J., Perli, R. and Sakellaris, P., 2005. Persistence of Business Cycles in Multisector RBC Models. Working Paper.

[6] Bullock, N. and Scholtes, S., 2009. US prime borrowers slip behind with payments as housing slump goes on. *The Financial Times*, August 5.

[7] Cai, J., 2005. Hopf bifurcation in the IS-LM business cycle model with time delay. *Electronic Journal of Differential Equations*, 15, 1-6.

[8] Chung, J., Guha, K. and Tett, G., 2008. Washington sends in cavalry to fight off full-blown crisis. *The Financial Times*, March 27.

[9] Grandmont, J-M., 1985. On Endogenous Competitive Business Cycles *Econometrica*, 53 (5), 995-1045.

[10] Mañé R., 1983. *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag.

[11] de Melo W. and Palis, J., 1982. *Geometric Theory of Dynamical Systems*, Springer-Verlag.

[12] Neri, U. and Venturi, B., 2007. Stability and bifurcations in IS-LM economic models *International Review of Economics*, 54 (1), 53-65.

[13] Scholtes, S., 2009. Commercial mortgages at risk of default five times higher in a year. *The Financial Times*, April 29.

[14] S & P Case-Shiller U.S. National Home Price Index Q1 2009, http://www2.standardandpoors.com/spf/pdf/index/csnational_values_052619.xl.

[15] IS/LM Model, Wikipedia, http://en.wikipedia.org/wiki/IS/LM_model.

[16] 1997 Asian Financial Crisis, Wikipedia, http://en.wikipedia.org/wiki/1997_Asian_Financial_Crisis.