Closed Form Formulas for Exotic Options and their Lifetime Distribution

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Abstract

We first recall the well known expression of the price of barrier options, and compute double barrier options by the mean of the iterated mirror principle. The formula for double barriers provides an intraday volatility estimator from the information of high-low-close prices. Then we give explicit formulas for the probability distribution function and the expectation of the exit time of single and double barrier options. These formulas allow to price time independent and time dependent rebates. They are also helpful to hedge barrier and double barrier options, when taking into account variations of the term structure of interest rates and of volatility. We also compute the price of rebates of double knock-out options that depend on which barrier is hit first, and of the BOOST, an option which pays the time spent in a corridor. All these formulas are either in closed form or double infinite series which converge like $e^{-\alpha n^2}$.

Key words : Double Barrier Options, Closed Forms, First Hitting Time.
JEL classification : G130.

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1. Introduction

Exit times of Ito processes from various domains appear to be a recurrent issue in option pricing and hedging. Some options explicitly involve their lifetime in their pay-off (time dependant rebates, etc.). Moreover, hedging — hence pricing — a non European option, the lifetime of which is thus uncertain, requires the knowledge of its lifetime distribution, in order to properly take into account the term structure of volatility and/or of interest rates.

In this paper, we first compute the value of a (single) barrier option, assuming constant interest rate and volatility, and the probability distribution of its lifetime. The method is a standard technique based on the reflection principle. Then we deduce the expected lifetime and its Laplace transform, which provides all its momenta. This gives a formula to price time independent and time dependent rebates, or rebates that are paid at a postponed date. Time independent rebates are also called "American digital options". Single barrier options are widely studied in the finance literature. See Rubinstein-Reiner [28] where prices of barrier options are computed, and Lamberton-Lapeyre [22, chap. 6] where the probability of passing a barrier is given as an exercise. Various books provide now the formula of single barrier options. See for instance Chesney and al. [10], Zhang [35] and Nelken [25]. Formulas for exit times distribution and expectation go back to the first half of the century, with Lévy [23] and Doob [12]. One of the best references concerning the exit time of Brownian and Ornstein-Uhlenbeck processes from variously shaped domains is Borodin-Salminen [7] (see also Prabhu [26]). An interesting history of the first passage time of a Brownian motion through a barrier can be found in Seshadri [33, Ch. 1].

In the second part, we compute the price of double barrier options and the probability distribution of their lifetime as a "quasi-closed form" formula : we use an iterated reflection principle, which again is not new (it goes back to the 50's), but provides extremely fast converging series, the n-th term being of the order of $\exp(-\alpha n^2)$. In practice, no more than 2 or 3 terms have a significant value. The series we get is different from that obtained by Geman and Yor [16] and it converges much faster. A methodology similar to ours is used in a recent paper by He and al. [20] to price double lookback options. From the double barrier option formula, we deduce an intraday volatility estimator using only the daily "high-low-close" information.

In the third part, we provide the value of rebates of double knock-out options. In order to provide the value of rebates that depend on which barrier is hit first, we also compute the distribution of the first hitting time of a barrier, conditionally to the fact that the other one has not been crossed before.

In the last part, we again give a quasi-closed form formula for the price of the *BOOST* option, using Laplace transforms. We recall that the *BOOST* can be considered as the rebate of a double knock-out option which is proportional to the number of days it stayed in (see sect. 6 for a precise definition). This option is described by Chesney and al. in their book [10].

2. Notations

Throughout the whole paper, $W_t = W(t,\omega)\,,\ (t,\omega) \in \mathbb{R}_+ \times \Omega$ will denote a standard Brownian motion under a given probability measure $\mathcal{P}$ on Ω. That is :

$$W_0 = 0 \qquad\qquad \operatorname{Var} W_t = t$$

The Gaussian distribution density and cumulative distribution, respectively g and N, are defined by :

$$g(x) = \frac{1}{\sqrt{2\pi}}\,\exp\left(-\frac{x^2}{2}\right) \qquad\qquad N(x) = \int_{-\infty}^{x} g(s)\,ds \tag{2.1}$$

The underlying of all the options considered here will be an asset S_t the price of which follows a "geometric Brownian motion" driven by the Brownian motion W_t. Letting μ denote the (constant) risk-neutral drift[1] and σ the (constant) volatility, we thus assume that S_t satisfies the diffusion equation :

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t \tag{2.2}$$

so that one has :

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\,W_t\right)$$

[1] That is, we assume that $\mathcal{P}$ is the risk-neutral probability and that $\mu = r_d - r_f$ where r_d is the domestic (currencies) or refinancing (equities) rate and r_f is the foreign (currencies) or dividend (equities) rate.

The reader is aware not to confuse the probability of the *intersection* of two events A and B :

$$\mathcal{P}(A \text{ and } B) = \mathcal{P}(A \cap B)$$

with the *conditional* probability of A upon the condition that B is realised :

$$\mathcal{P}(A \,|\, B) = \frac{\mathcal{P}(A \cap B)}{\mathcal{P}(B)}$$

The acronym P.D.F. stands for "Probability Density Function", while P.D.E. means "Partial Differential Equation". The P.D.F. $\psi_A(x)$ of the random variable X *constrained by the event* A is defined by :

$$\mathcal{P}\left(X \in [x, x + dx] \text{ and } A\right) = \psi_A(x)\, dx$$

In particular, one has :

$$\int_{-\infty}^{\infty} \psi_A(x)\, dx = \mathcal{P}(A)$$

Generally speaking, we reserve the letter ψ for P.D.F. with respect to the x variable, while φ will be used for those with respect to the t variable. The letter τ will always denote a stochastic hitting time.

In all closed form formulas provided here for option prices, the option nominal is set to 1.

3. Single Barriers

The results of this section are not new[2], but they will be useful for the next one. We adopt Lamberton-Lapeyre's methodology [22] based on the reflection principle (see also Revuz-Yor [29, p.101]).

3.1. The Mirror Principle for Drift-free Processes

This "trick" allows one to compute the probability of crossing the barrier. Let $h > 0$ be the barrier level and T be the time horizon. We define the process W'_t

[2]They were known already in the 70's by finance people (Merton [24]), and in the 50's (and perhaps even before) by probabilists (Levy [23]).

by :

$$W'_t = W_t \qquad \text{if} \quad W_s < h \quad \forall s < t$$

$$W'_t = 2h - W_t \qquad \text{if} \quad \exists s < t\,, \quad W_s = h$$

The process W'_t is again a standard Brownian motion (it is a martingale with independent increments and volatility 1). It has the following property :

$$\forall T > 0 \qquad \left(W_T = x < h \text{ and } \max_{[0,T]} W_t \geq h \right) \iff W'_T = 2h - x \tag{3.1}$$

Therefore, $\max_{[0,T]} W_t \geq h$ if, and only if $W_T \geq h$ or $W'_T \geq h$, and these two events are incompatible. As a consequence, we may add their probabilities :

$$\mathcal{P}\left(\max_{[0,T]} W_t \geq h \right) = 2\,\mathcal{P}\left(W_T \geq h\right) = 2\,N\!\left(-\frac{h}{\sqrt{T}} \right) \tag{3.2}$$

Let τ_h be the first time at which W_t hits the barrier h :

$$\tau_h = \min\left\{ t > 0,\ W_t = h \right\} \quad (\text{possibly } > T \text{ or even } = \infty\,)$$

From (3.2), we deduce that :

$$\mathcal{P}(\tau_h \leq T) = 2\,N\!\left(-\frac{h}{\sqrt{T}} \right) \tag{3.3}$$

This equation provides, by a simple derivation with respect to t, the P.D.F. of τ_h :

$$\varphi_h(t) = \frac{\partial}{\partial T}\mathcal{P}(\tau_h \leq T) = \frac{h}{\sqrt{2\pi t^3}}\ \exp\left(-\frac{h^2}{2t} \right)$$

3.2. Processes with Drift

The case of processes with a constant drift will be solved by the mean of Girsanov theorem. Let S_t be the price at time t of the underlying asset. We set :

$$X_t = \frac{1}{\sigma}\,\log\frac{S_t}{S_0} = \lambda\,t + W_t \qquad\qquad \lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2}$$

Thanks to Girsanov theorem[3], X_t is a standard Brownian motion under the probability $\mathcal{P}^\lambda$ which has the following density with respect to $\mathcal{P}$:

$$\frac{d\mathcal{P}^\lambda}{d\mathcal{P}}|_t = \exp\left(-\lambda X_t + \frac{\lambda^2 t}{2}\right)$$

We now know, owing to (3.1), that, if [4]$x < h$:

$$\mathcal{P}^\lambda\left(X_T \in [x, x+dx] \text{ and } \max_{[0,T]} X_t < h\right) = \frac{dx}{\sqrt{2\pi T}}\left(\exp\left(-\frac{x^2}{2T}\right) - \exp\left(-\frac{(2h-x)^2}{2T}\right)\right)$$

hence

$$\mathcal{P}\left(X_T \in [x, x+dx] \text{ and } \max_{[0,T]} X_t < h\right) = \psi_h^\lambda(x,T)\,dx \tag{3.4}$$

with

$$\psi_h^\lambda(x,T) = \frac{1}{\sqrt{2\pi T}}\left(\exp\left(-\frac{(x-\lambda T)^2}{2T}\right) - \exp\left(2\lambda h - \frac{(2h+\lambda T - x)^2}{2T}\right)\right) \tag{3.5}$$

In particular :

$$\begin{aligned}\mathcal{P}\left(\max_{[0,T]} x_t < h\right) &= \int_{-\infty}^{h} \psi_h^\lambda(x,T)\,dx \\ &= N\left(\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right) - e^{2\lambda h}\, N\left(-\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right) \qquad (3.6)\end{aligned}$$

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[3]We use here a very simple form of this theorem, and one may also apply (again in a simple form) Cameron-Martin formula:

$$\mathcal{P}^\lambda\left(X_t \in [x, x+dx]\right) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)dx \qquad \mathcal{P}\left(X_t \in [x, x+dx]\right) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-\lambda t)^2}{2t}\right)dx$$

thus:

$$\frac{d\mathcal{P}^\lambda}{d\mathcal{P}} = \exp\left(\frac{(x-\lambda t)^2 - x^2}{2t}\right) = \exp\left(-\lambda x + \frac{\lambda^2 t}{2}\right)$$

[4]In the sequel, dx is meant to be an infinitesimal length of interval. Equalities should be understood as valid at the limit $dx \to 0$, when divided by dx :

$$\mathcal{P}\left(X \in [x, x+dx]\right) = \psi(x)\,dx \quad \Longleftrightarrow \quad \lim_{dx\to 0}\frac{1}{dx}\mathcal{P}\left(X \in [x, x+dx]\right) = \psi(x)$$

Let τ_h^λ be the first time X_t hits the barrier h (still assumed positive). We know with certainty that $X_{\tau_h^\lambda} = h$. Therefore, the P.D.F. (under probability $\mathcal{P}$) of τ_h^λ is given by :

$$\varphi_h^\lambda(t) = \frac{d\mathcal{P}}{d\mathcal{P}^\lambda}|_{X_t=h}\, \varphi_h(t) = \frac{h}{\sqrt{2\pi t^3}}\, \exp\left(\lambda h - \frac{\lambda^2 t}{2} - \frac{h^2}{2t}\right) \tag{3.7}$$

One can easily check that :

$$\varphi_h^\lambda(T) = -\frac{d}{dT}\mathcal{P}\left(\max_{[0,T]} x_t < h\right) \tag{3.8}$$

3.3. Price of (Single) Barrier Options

We here consider only knock-in and knock-out options that can knock in or out all along their life, like currency options. We exclude those for which the barrier is only active at expiration ("European" barrier), which are actually combinations of binary options and vanilla ones, like limited caps, etc., and those barriers that apply in a "time window". For options which only check wether the closing price is beyond the barrier, we refer the reader to Broadie-Glasserman-Kou [5] and [6].

Let H denote an upper barrier and L a lower one. The following relation always holds :

$$\text{Knock-In} \quad + \quad \text{Knock-Out} \quad = \quad \text{Vanilla}$$

provided they all have same strike, same maturity and (except for the vanilla one) same barrier. We shall therefore compute only knock-out options. The price of an "Up-and-Out Call" will be $\mathrm{UOC}(S,K,H)$, while that of an "Down-and-Out Put" will be $\mathrm{DOP}(S,K,L)$, etc. There is no special relation between Puts and Calls[5]. The price of an Up-and-Out option with trigger H, the pay-off of which at the expiration date T is $f(S_T)$, is provided by the knowledge of the function ψ_h^λ :

$$\mathrm{UO}_f(S_0,H) = e^{-rT}\int_{-\infty}^{h} \psi_h^\lambda(x,T)\, f\left(S_0 \exp(\sigma x)\right) dx$$

[5]Some approximations are available for "symmetric like" structures, for instance $P(S,K,H) \simeq \mathrm{cst}.C(S,K',L)$ with $K' = S^2/K$ and $L = S^2/H$. The fail from exact equality comes from interest rates and from a possible volatility smile (which is not taken into account in this study). See [9] and [15].

where $h = \dfrac{1}{\sigma} \log \dfrac{H}{S_0}$. Replacing the process X_t by $-X_t$ yields the price of Down-and-Out options.

In the following formulae, $r = r^d$ denotes the domestic (refinancing) interest rate, and μ is the risk-neutral drift of S_t that is $\mu = r^d - r^f$, where r^f is the dividend rate earned by the underlying (or foreign interest rate for a currency).

For easier notations, we set $S_0 = S$ and :

$$h = \frac{1}{\sigma} \log \frac{H}{S} \qquad k = \frac{1}{\sigma} \log \frac{K}{S} \qquad \ell = \frac{1}{\sigma} \log \frac{L}{S}$$

$$\lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2} \qquad \lambda' = \frac{\mu}{\sigma} + \frac{\sigma}{2}$$

For Up-and-Out options :

$$\alpha = e^{2\lambda h} \qquad \alpha' = e^{2\lambda' h} = \frac{\alpha H^2}{S^2}$$

$$\begin{cases} d_1 = \lambda' \sqrt{T} - k/\sqrt{T} \\ d_2 = \lambda \sqrt{T} - k/\sqrt{T} \\ d_3 = \lambda' \sqrt{T} - h/\sqrt{T} \\ d_4 = \lambda \sqrt{T} - h/\sqrt{T} \end{cases} \qquad \begin{cases} d_5 = -\lambda \sqrt{T} - h/\sqrt{T} \\ d_6 = -\lambda' \sqrt{T} - h/\sqrt{T} \\ d_7 = -\lambda \sqrt{T} - (2h - k)/\sqrt{T} \\ d_8 = -\lambda' \sqrt{T} - (2h - k)/\sqrt{T} \end{cases}$$

For Down-and-Out options :

$$\alpha = e^{2\lambda \ell} \qquad \alpha' = e^{2\lambda' \ell} = \frac{\alpha L^2}{S^2}$$

$$\begin{cases} d_1 = \lambda' \sqrt{T} - k/\sqrt{T} \\ d_2 = \lambda \sqrt{T} - k/\sqrt{T} \\ d_3 = \lambda' \sqrt{T} - \ell/\sqrt{T} \\ d_4 = \lambda \sqrt{T} - \ell/\sqrt{T} \end{cases} \qquad \begin{cases} d_5 = -\lambda \sqrt{T} - \ell/\sqrt{T} \\ d_6 = -\lambda' \sqrt{T} - \ell/\sqrt{T} \\ d_7 = -\lambda \sqrt{T} - (2\ell - k)/\sqrt{T} \\ d_8 = -\lambda' \sqrt{T} - (2\ell - k)/\sqrt{T} \end{cases}$$

For simplicity, we denote $N_i = N(d_i)$. Up-and-Out Calls and Down-and-Out Puts (with $L < K < H$ otherwise these are void options) are given by :

$$\mathrm{UOC}(S, K, H) = e^{(\mu - r)T} S \left(N_1 - N_3 - \alpha' \left(N_6 - N_8 \right) \right)$$

$$- e^{-rT} K \left(N_2 - N_4 - \alpha \left(N_5 - N_7 \right) \right)$$

$$\mathrm{DOP}(S,K,L) = e^{-rT} K \left(N_4 - N_2 - \alpha \left(N_7 - N_5 \right) \right)$$

$$- e^{(\mu - r)T} S \left(N_3 - N_1 - \alpha' \left(N_8 - N_6 \right) \right)$$

Down-and-Out Calls and Up-and-Out Puts depend on the location of the trigger with respect to the strike :

$$\mathrm{DOC}(S,K,L \leq K) = e^{(\mu - r)T} S \left(N_1 - \alpha' \left(1 - N_8 \right) \right) - e^{-rT} K \left(N_2 - \alpha \left(1 - N_7 \right) \right)$$

$$\mathrm{DOC}(S,K,L \geq K) = e^{(\mu - r)T} S \left(N_3 - \alpha' \left(1 - N_6 \right) \right) - e^{-rT} K \left(N_4 - \alpha \left(1 - N_5 \right) \right)$$

$$\mathrm{UOP}(S,K,H \geq K) = e^{-rT} K \left(1 - N_2 - \alpha N_7 \right) - e^{(\mu - r)T} S \left(1 - N_1 - \alpha' N_8 \right)$$

$$\mathrm{UOP}(S,K,H \leq K) = e^{-rT} K \left(1 - N_4 - \alpha N_5 \right) - e^{(\mu - r)T} S \left(1 - N_3 - \alpha' N_6 \right)$$

Remark 1. *These formulae should only be considered as a basis to price barrier options. Not only do they take into account flat and non stochastic interest rate and volatility, but even in this case barrier options involve violent discontinuities and uneven behavior with respect to volatility changes, that make them a real challenge to hedge. The impact of transaction costs will depend on the size of the book. We also warn the reader of the unbounded delta Δ of "irregular" options*[6] *which makes a pure "Black-Scholes" dynamic hedging strategy inapplicable, because of "liquidity holes", and causes irreducible risk*[7]*. Hedging difficulties are thoroughly analysed in Taleb [34] and an attempt to optimise the sharing of the hedge between static and dynamic is discussed in Avellaneda–Levy-Paras [3].*

[6] $\mathrm{DOC}(S,K,L \leq K)$ and $\mathrm{UOP}(S,K,H \geq K)$ are "regular" knock-out options, but $\mathrm{DOC}(S,K,L \geq K)$ and $\mathrm{UOP}(S,K,H \leq K)$ are irregular : the trigger is in the money and they contain an "American bet" which causes hedging difficulties close to maturity. The same holds for the "reverse knock-out" $\mathrm{UOC}(S,K,H)$ and $\mathrm{DOP}(S,K,L)$.

[7] I remember a trader giving an order to his broker : "Sell 100,000 shares !" and the broker : "But to whom, sir ?".

3.4. Rebates and American Digitals

Often, knock-out options include a rebate delivered to the buyer as a compensation when the underlying hits the barrier. If the rebate is paid just after knocking out, its price per unit face value equals the expectation of the discount factor, under the condition that the underlying crosses the barrier. If, independently of the date the option is knocked out, the rebate is paid at expiry, then its price is simply equal to the probability of hitting the barrier, discounted by e^{-rT}, where r is the rate applicable on this period. In this latter case, the Up-and-Out Rebate is given by eqn.(3.6) :

$$\mathrm{UOR}_{\mathrm{end}}(S,K,H) = e^{-rT}\left(1-\mathcal{P}\left(\max_{[0,T]} X_t < h\right)\right)$$

$$= e^{-rT}\,N\!\left(\lambda\sqrt{T}-\frac{h}{\sqrt{T}}\right)+e^{2\lambda h-rT}\,N\!\left(-\frac{h}{\sqrt{T}}-\lambda\sqrt{T}\right) \quad (3.9)$$

and the Down-and-Out Rebate, by changing the sign of X_t :

$$\mathrm{DOR}_{\mathrm{end}}(S,K,L) = e^{-rT}\,N\!\left(\frac{\ell}{\sqrt{T}}-\lambda\sqrt{T}\right)+e^{2\lambda\ell-rT}\,N\!\left(\frac{\ell}{\sqrt{T}}+\lambda\sqrt{T}\right) \quad (3.10)$$

When the rebate is paid just after knocking out, and still assuming constant interest rate, one has :

$$\mathrm{UOR}_{\mathrm{KO}}(S,K,H) = \int_0^T e^{-rt}\varphi_h^{\lambda}(t)\,dt$$

$$= e^{(\lambda-\sqrt{\lambda^2+2r})h}\,N\!\left(\sqrt{(\lambda^2+2r)\,T}-\frac{h}{\sqrt{T}}\right)+e^{(\lambda+\sqrt{\lambda^2+2r})h}\,N\!\left(-\frac{h}{\sqrt{T}}-\sqrt{(\lambda^2+2r)\,T}\right)$$

and

$$\mathrm{DOR}_{\mathrm{KO}}(S,K,L) = \int_0^T e^{-rt}\varphi_{-\ell}^{-\lambda}(t)\,dt$$

$$= e^{(\lambda+\sqrt{\lambda^2+2r})\ell}\,N\!\left(\sqrt{(\lambda^2+2r)\,T}+\frac{\ell}{\sqrt{T}}\right)+e^{(\lambda-\sqrt{\lambda^2+2r})\ell}\,N\!\left(\frac{\ell}{\sqrt{T}}-\sqrt{(\lambda^2+2r)\,T}\right) \quad (3.11)$$

Remark 2. *When taking $T=+\infty$, the rebate price provides the Laplace transform of the exit time distribution (considering the interest rate r as the variable) :*

$$\int_0^{\infty} e^{-rt}\varphi_h^{\lambda}(t)\,dt = \exp\left(\left(\lambda-\sqrt{\lambda^2+2r}\right)h\right)$$

One knows that the coefficients of the Taylor expansion of the Laplace transform at 0 provide the momenta of the distribution.

Remark 3. *The price of a constant rebate is also the value of the "American digital" option which pays a fixed amount as soon as the spot crosses a given trigger.*

3.5. Delta of (Single) Barrier Options

From the formulas giving the prices of barrier options, one can easily deduce, through a simple differentiation with respect to the spot, the value of the Δ. Again, we compute them only for knock-out options (g is the standard Gaussian P.D.F.) :

$$\begin{aligned}\Delta_{\mathrm{UOC}} &= e^{(\mu-r)T}\left(N_1 - N_3 + 2\alpha'\frac{r}{\sigma^2}\left(N_6 - N_8\right)\right) \\ &\quad - 2\,e^{-rT}\left(\alpha\,\frac{\lambda K}{\sigma S}\left(N_5 - N_7\right) + \frac{H-K}{S\sigma\sqrt{T}}\,g(d_4)\right)\end{aligned}$$

For symmetry reasons, the Δ of the Down-and-Out Put has the same expression, for both options are the two sides of the same analytic solution of Black-Scholes P.D.E., vanishing along the barrier :

$$\begin{aligned}\Delta_{\mathrm{DOP}} &= e^{(\mu-r)T}\left(N_1 - N_3 + 2\alpha'\frac{r}{\sigma^2}\left(N_6 - N_8\right)\right) \\ &\quad - 2\,e^{-rT}\left(\alpha\,\frac{\lambda K}{\sigma S}\left(N_5 - N_7\right) - \frac{K-L}{S\sigma\sqrt{T}}\,g(d_4)\right)\end{aligned}$$

Like previously for prices, Down-and-Out Calls and Up-and-Out Puts deltas depend on wether the strike is above or below the barrier :

$$\Delta_{\mathrm{DOC},K\geq L} = e^{(\mu-r)T}\left(N_1 + 2\,\frac{r}{\sigma^2}\,\alpha'\left(1 - N_8\right)\right) - 2\,e^{-rT}\,\frac{\lambda K}{\sigma S}\,\alpha\left(1 - N_7\right)$$

$$\begin{aligned}\Delta_{\mathrm{DOC},K\leq L} &= e^{(\mu-r)T}\left(N_3 + 2\,\frac{r}{\sigma^2}\,\alpha'\left(1 - N_6\right)\right) - 2\,e^{-rT}\,\frac{\lambda K}{\sigma S}\,\alpha\left(1 - N_5\right) \\ &\quad + 2\,e^{-rT}\,\frac{L-K}{S\sigma\sqrt{T}}\,g(d_4)\end{aligned}$$

$$\Delta_{\mathrm{UOP},K\leq H} = 2\,e^{-rT}\,\frac{\lambda K}{\sigma S}\,\alpha N_7 - e^{(\mu-r)T}\left(1 - N_1 + 2\,\frac{r}{\sigma^2}\,\alpha' N_8\right)$$

$$\Delta_{\mathrm{UOP},K\geq H} = 2\,e^{-rT}\,\frac{\lambda K}{\sigma S}\,\alpha N_5 - e^{(\mu-r)T}\left(1 - N_3 + 2\,\frac{r}{\sigma^2}\,\alpha' N_6\right)$$
$$- 2\,e^{-rT}\,\frac{K-H}{S\sigma\sqrt{T}}\,g(d_4)$$

Remark 4. *Note that the term $g(d_4)$ appears when the option pay-off presents a discontinuity at the barrier. When the option is a regular knock out — like a Down-and-Out Call with a strike above the barrier — this term cancels out. The same cancellation occurs in the Δ of standard Calls and Puts, but not in that of binary options.*

3.6. Expectation of the Exit Time

The distribution of the exit time $\varphi_h^\lambda(t)$ is information worthy of note. Indeed, in a stochastic interest rate framework, it provides an approximation of the hedge splitting over the different forwards of the underlying which is widely adopted by traders (in particular in the commodity market). In this case, the sensitivity of a barrier option to one particular forward price (at a date before the option maturity) is equal to the probability density of knocking out at this date multiplied by the Δ of the option on the barrier at this precise date. Therefore, one can deduce how to display the hedge over the various maturities. In a first attempt, one can adjust the "duration" of this distribution (its definition is a straightforward extent of Macauley's one for bonds) to the expectation of the exit time, or better, to the conditional expectation of the exit time assuming the option is knocked out, the hedge being dispatched between a combination of forward contracts that globally has this duration and an amount of forward contracts at maturity, according to the probability of remaining until expiration, given by eqn.(3.6). We here provide the expectation of the exit time, unconditional or conditioned by a maximum maturity T :

$$\mathbf{E}(\tau_{h,T}^\lambda) = \frac{h}{\lambda} - e^{2\lambda h}\left(T + \frac{h}{\lambda}\right) N\left(-\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right) + \left(T - \frac{h}{\lambda}\right) N\left(\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right)$$

$$\mathbf{E}\left(\tau_h^\lambda \,|\tau_h^\lambda < T\right) = \frac{h\left(N\left(\lambda\sqrt{T} - \frac{h}{\sqrt{T}}\right) - e^{2\lambda h} N\left(-\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right)\right)}{\lambda\left(N\left(\lambda\sqrt{T} - \frac{h}{\sqrt{T}}\right) + e^{2\lambda h}\, N\left(-\frac{h}{\sqrt{T}} - \lambda\sqrt{T}\right)\right)}$$

Remark 5. *Note that the unconditional expectation of the exit time has the simple expression:*

$$\mathbf{E}(\tau_h^\lambda) = \frac{h}{\lambda}$$

It tends to $+\infty$ *as* λ *tends to 0.*

Let $\Delta_H(t)$ be the forward value of the option Δ at date t assuming the spot hits the barrier at this time $(S_t = H)$. The distribution of hedges will, up to the first order, depend on the expectation :

$$\mathbf{E}\left(\tau_h^\lambda \; \Delta_H(\tau_h^\lambda) \,|\, \tau_h^\lambda < T\right)$$

Let $\pi^* = \mathcal{P}\left(\max_{[0,T]} X_t < h\right)$ be the probability of the option not being knocked-out. One has :

$$\mathbf{E}\left(\tau_h^\lambda \; \Delta_H(\tau_h^\lambda) \,|\, \tau_h^\lambda < T\right) = \frac{1}{1-\pi^*} \int_0^T t\, \Delta_H(t)\, \varphi_h^\lambda(t)\, dt$$

This integral cannot, in general, be computed in closed form, but standard numerical techniques like Gauss-Legendre (see [27]) allow very fast and accurate evaluations of a one-dimensional integral.

This formula should be used as follows : assume that hedging the barrier option requires one to buy an amount Δ of underlying (spot) in a constant interest rate framework. Then, when interest rates vary, one should split Δ between an amount $\pi^*\Delta/B(0,T)$ of forward contracts on the underlying at date T (where $B(0,T) = e^{-rT}$ is the discount factor on the interval $[0,T]$) and quantities $\delta(t_i)$ of forward contracts at dates $t_1, \ldots, t_n < T$ such that :

$$\sum \delta(t_i) B(0,t_i) = (1-\pi^*)\,\Delta$$

and

$$\frac{\sum \delta(t_i) B(0,t_i) \; \Delta_H(t_i)\, t_i}{(1-\pi^*)\,\Delta} = \mathbf{E}\left(\tau_h^\lambda\, \Delta_H(\tau_h^\lambda) \,|\, \tau_h^\lambda < T\right)$$

Such a hedge will at least protect against a parallel shift of the yield curve (in case of a stock) and both yield curves (in case of a currency).

4. Double Barriers

4.1. The Iterated Mirror Principle

Let us first compute the probability that a standard (drift-free) Brownian motion W_t remains in a corridor $[\ell, h]$ (with $\ell < 0 < h$) during the period $[0, T]$. Denote by $\delta = h - \ell$ the corridor width and by $\tau_{h,\ell}$ the first time W_t hits one of the two barriers (possibly $\tau_{h,\ell} > T$). We also define:

$$W_\tau^T = \begin{cases} W_{\tau_{h,\ell}} \quad (= h \text{ or } \ell) \quad \text{if} \quad \tau_{h,\ell} < T \\ W_T \qquad \text{if} \quad \tau_{h,\ell} \geq T \end{cases}$$

According to the mirror principle, one has[8]:

$$\begin{aligned} \mathcal{P}\left(W_T = x \text{ and } \tau_{h,\ell} \geq T\right) &= \mathcal{P}\left(W_T = x\right) - \mathcal{P}\left(W_T = x \text{ and } W_\tau^T = h\right) \\ &\quad - \mathcal{P}\left(W_T = x \text{ and } W_\tau^T = \ell\right) \\ &= \mathcal{P}\left(W_T = x\right) - \mathcal{P}\left(W_T = 2h - x \text{ and } W_\tau^T = h\right) \\ &\quad - \mathcal{P}\left(W_T = 2\ell - x \text{ and } W_\tau^T = \ell\right) \end{aligned}$$

Then

$$\begin{aligned} \mathcal{P}\left(W_T = 2h - x \text{ and } W_\tau^T = h\right) &= \mathcal{P}\left(W_T = 2h - x\right) \\ &\quad - \mathcal{P}\left(W_T = 2h - x \text{ and } W_\tau^T = \ell\right) \end{aligned}$$

$$= \mathcal{P}\left(W_T = 2h - x\right) - \mathcal{P}\left(W_T = x - 2\delta \text{ and } W_\tau^T = \ell\right)$$

We end into the following expression for the distribution density of W_T under the condition that W_t remains inside the corridor for all $t \in [0, T]$:

$$\psi_{h,\ell,T}(x) = \lim_{dx \to 0} \frac{1}{dx} \mathcal{P}\left(W_T \in [x, x + dx] \text{ and } \tau_{h,\ell} \geq T\right)$$

[8]Here, we write for simplicity $W_T = x$ for $W_T \in [x, x + dx]$.

$$= \mathbf{1}_{[\ell,h]} \sum_{n=-\infty}^{+\infty} \left(g_T(x+2n\delta) - g_T(2h-x+2n\delta)\right) \tag{4.1}$$

where $g_T(x)$ denotes the centered Gaussian distribution density of standard deviation $\sqrt{T}$:

$$g_T(x) = \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{x^2}{2T}\right)$$

One may easily check that $\psi_{h,\ell,T}(\ell) = \psi_{h,\ell,T}(h) = 0$.

4.2. Double Knock-Out Calls

The notations are the same as those for single barriers :

$$h = \frac{1}{\sigma} \log \frac{H}{S} \qquad k = \frac{1}{\sigma} \log \frac{K}{S} \qquad \ell = \frac{1}{\sigma} \log \frac{L}{S}$$

$$\lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2} \qquad \lambda' = \frac{\mu}{\sigma} + \frac{\sigma}{2} \qquad \delta = h - \ell$$

We first assume that the strike K is between the barriers L and H. Then the price of the Double Knock-Out Call is :

$$\mathrm{DKOC}(S, K, L, H) = e^{-rT} \int_k^h (S\, e^{\sigma x} - K)\, \psi^{\lambda}_{h,\ell,T}(x)\, dx$$

where the distribution density $\psi^{\lambda}_{h,\ell,T}(x)$ is given by Girsanov theorem :

$$\psi^{\lambda}_{h,\ell,T}(x) = e^{-\frac{1}{2}\lambda^2 T + \lambda x} \psi_{h,\ell,T}(x) \tag{4.2}$$

Let us set, for any $n \in \mathbb{Z}$:

$$\begin{aligned} I_n(u,k,h,\delta) &= \int_k^h e^{-\frac{1}{2}u^2 T + ux}\, g_T(x+2n\delta)\, dx \\ &= e^{-2nu\delta} \int_k^h g_T(x+2n\delta-u)\, dx \\ &= e^{-2nu\delta} \left(N\left(\frac{h+2n\delta}{\sqrt{T}} - u\sqrt{T}\right) - N\left(\frac{k+2n\delta}{\sqrt{T}} - u\sqrt{T}\right)\right) \end{aligned}$$

and

$$J_n(u,k,h,\delta) = \int_k^h e^{-\frac{1}{2}u^2T+ux}\, g_T(2h-x+2n\delta)\,dx$$
$$= e^{2u(n\delta+h)}\left(N\left(\frac{2h-k+2n\delta}{\sqrt{T}}+u\sqrt{T}\right)-N\left(\frac{h+2n\delta}{\sqrt{T}}+u\sqrt{T}\right)\right)$$

These series tend towards 0 like a Gaussian : for reasonable values of K , L and H , they numerically vanish for $|n| \geq 4$, and one has :

$$\mathrm{DKOC}(S,K,L,H) = e^{(\mu-r)T}\, S \sum_{n=-\infty}^{\infty} \left(I_n(\lambda',k,h,\delta) - J_n(\lambda',k,h,\delta)\right)$$
$$- e^{-rT}\, K \sum_{n=-\infty}^{\infty} \left(I_n(\lambda,k,h,\delta) - J_n(\lambda,k,h,\delta)\right)$$

If $K < L$, then the lower bound of the integrals must be set to ℓ and the formula becomes :

$$\mathrm{DKOC}(S,K,L,H) = e^{(\mu-r)T}\, S \sum_{n=-\infty}^{\infty} \left(I_n(\lambda',\ell,h,\delta) - J_n(\lambda',\ell,h,\delta)\right)$$
$$- e^{-rT}\, K \sum_{n=-\infty}^{\infty} \left(I_n(\lambda,\ell,h,\delta) - J_n(\lambda,\ell,h,\delta)\right)$$

with

$$I_n(u,\ell,h,\delta) = \int_\ell^h e^{-\frac{1}{2}u^2T+ux}\, g_T(x+2n\delta)\,dx$$
$$= e^{-2nu\delta}\left(N\left(\frac{h+2n\delta}{} - u\sqrt{T}\right)-N\left(\frac{h+(2n-1)\delta}{\sqrt{T}}-u\sqrt{T}\right)\right)$$

and

$$J_n(u,\ell,h,\delta) = \int_\ell^h e^{-\frac{1}{2}u^2T+ux}\, g_T(2h-x+2n\delta)\,dx$$
$$= e^{2u(n\delta+h)}\left(N\left(\frac{h+(2n+1)\delta}{\sqrt{T}}+u\sqrt{T}\right)-N\left(\frac{h+2n\delta}{\sqrt{T}}+u\sqrt{T}\right)\right)$$

4.3. Double Knock-Out Puts

As for the Calls, the price of a Double Knock-Out Puts depends on the upper barrier setting with respect to the strike. One has :

$$\begin{aligned}
\mathrm{DKOP}(S,K,L,H) &= e^{-rT}\,K\sum_{n=-\infty}^{\infty}\left(I_n(\lambda,\ell,k,\delta)-J_n(\lambda,\ell,k,\delta)\right)\\
&\quad - e^{(\mu-r)T}\,S\sum_{n=-\infty}^{\infty}\left(I_n(\lambda',\ell,k,\delta)-J_n(\lambda',\ell,k,\delta)\right)
\end{aligned}$$

where, if $L<K<H$:

$$\begin{aligned}
I_n(u,\ell,k,\delta) &= \int_\ell^k e^{-\frac{1}{2}u^2T+ux}\,g_T(x+2n\delta)\,dx\\
&= e^{-2nu\delta}\left(N\left(\frac{k+2n\delta}{\sqrt{T}}-u\sqrt{T}\right)-N\left(\frac{h+(2n-1)\delta}{\sqrt{T}}-u\sqrt{T}\right)\right)\\
J_n(u,\ell,k,\delta) &= \int_\ell^k e^{-\frac{1}{2}u^2T+ux}\,g_T(2h-x+2n\delta)\,dx\\
&= e^{2u(n\delta+h)}\left(N\left(\frac{h+(2n+1)\delta}{\sqrt{T}}+u\sqrt{T}\right)-N\left(\frac{2h-k+2n\delta}{\sqrt{T}}+u\sqrt{T}\right)\right)
\end{aligned}$$

and, if $K\geq H$:

$$\begin{aligned}
\mathrm{DKOP}(S,K,L,H) &= e^{-rT}\,K\sum_{n=-\infty}^{\infty}\left(I_n(\lambda,\ell,h,\delta)-J_n(\lambda,\ell,h,\delta)\right)\\
&\quad - e^{(\mu-r)T}\,S\sum_{n=-\infty}^{\infty}\left(I_n(\lambda',\ell,h,\delta)-J_n(\lambda',\ell,h,\delta)\right)
\end{aligned}$$

Remark 6. *The price of barrier, and even more double barrier options is very sensitive to the term structure of interest rates and to the shape of the volatility surface (the "smile"). If one needs to take them into account, then, generally speaking, no closed form formula is available but, because of the Dirichlet boundary conditions, the value of the option is efficiently computed through a θ-scheme with meshes precisely on the barriers (see [27]).*

4.4. Intraday Volatility Estimator

Data bases are usually daily settled, but they provide a very useful information : the highest quotation of the day, the lowest, the opening and the closing. One can apply the present computations to deduce the joint distribution of the triple $\left(\max_{t\le T} S_t\,,\min_{t\le T} S_t\,,\,S_T\right)$ and therefore an estimator of the intraday volatility. Such an estimation may, for instance, allow one to optimise the frequency of dynamic hedging.

As previously, we assume that :

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW$$

so that :

$$\begin{aligned} S_t &= S_0 \exp \sigma X_t \\ X_t &= \lambda t + W_t \qquad\qquad \lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2} \end{aligned}$$

Let $T > 0$. One has :

$$\mathcal{P}\left(X_T \in [x, x+dx] \text{ and } \max_{t\le T} X_t \le h\,,\,\min_{t\le T} X_t \ge \ell\right) = \psi^{\lambda}_{h,\ell,T}(x)\,dx$$

thus the joint P.D.F. of $\left(\max_{t\le T} X_t\,,\min_{t\le T} X_t\,,\,X_T\right)$ is :

$$\begin{aligned} \frac{\partial^2}{\partial h \partial \ell}\psi^{\lambda}_{h,\ell,T}(x) &= e^{-\frac{\lambda^2 T}{2}+\lambda x}\,\frac{\partial^2}{\partial h \partial \ell}\psi_{h,\ell,T}(x) \\ &= \mathbf{1}_{\ell<x<h}\, e^{-\frac{\lambda^2 T}{2}+\lambda x} \sum_{n=-\infty}^{+\infty} \frac{\partial^2}{\partial h\partial \ell}\left(g_T(x+2n\delta) - g_T(2h-x+2n\delta)\right) \\ &= \mathbf{1}_{\ell<x<h}\, e^{-\frac{\lambda^2 T}{2}+\lambda x} \sum_{n=-\infty}^{+\infty} 4n\left((n+1)\,g''_T(2h-x+2n\delta) - n g''_T(x+2n\delta)\right) \quad (4.3) \end{aligned}$$

where

$$g_T''(x) = \frac{d^2}{dx^2}\, g_T(x) = \frac{x^2 - T}{\sqrt{2\pi T^5}}\ \exp\left(-\frac{x^2}{2T}\right)$$

Given a time series :

$$(S_{\text{open}}(t_i)\ , S_{\text{high}}(t_i)\ , S_{\text{low}}(t_i)\ , S_{\text{close}}(t_i)) \qquad\qquad i = 1, \ldots, N$$

one computes :

$$h_i = \log \frac{S_{\text{high}}(t_i)}{S_{\text{open}}(t_i)} \qquad\qquad \ell_i = \log \frac{S_{\text{low}}(t_i)}{S_{\text{open}}(t_i)} \qquad\qquad x_i = \log \frac{S_{\text{close}}(t_i)}{S_{\text{open}}(t_i)}$$

and finds the maximum likelihood values of λ and σ such that the P.D.F. of $(h_i/\sigma\ , \ell_i/\sigma\ , x_i/\sigma)$ be given by formula $(4.3)_\lambda$.

Remark 7. *One may also assume that λ is given as the risk-neutral drift and perform the maximum likelihood estimation only on σ , but in a market with a strong trend, this might lead to hedging errors.*

5. Corridors and Rebates of Double Barrier Options

5.1. Corridors

A *Corridor* is an option that pays a fixed amount at a given date, provided the underlying remained within a range $[L, H]$ all the time until expiration. According to the previous analysis, its price is given by :

$$\text{Corridor}(S, L, H, T) = e^{-rT} \sum_{n=-\infty}^{\infty} (I_n(\lambda, \ell, h, \delta) - J_n(\lambda, \ell, h, \delta)) \tag{5.1}$$

Remark 8. *The expression giving the price of a Corridor corresponds to the term in K in the Double Knock-Out Calls and Puts, when K is outside the range $[L, H]$. An intuitive way to understand this is to see a Corridor as an extreme Double Knock-Out Put with an infinitely high strike and an infinitely small amount accordingly, or to differentiate the option value with respect to K .*

5.2. Symmetric Rebates

If a rebate is included in a double knock-out option, then in the case it is paid at the expiration date T, whenever the option is knocked out, it is formally equivalent to the complement of a Corridor option to the zero-coupon of maturity T. The price (per unit) of a Rebate Double Knock-Out option, paid at the end, is thus given by :

$$\mathrm{RDKO_{end}}(S,L,H,T) = e^{-rT}\left(1 - \sum_{n=-\infty}^{\infty}\left(I_n(\lambda,\ell,h,\delta) - J_n(\lambda,\ell,h,\delta)\right)\right)$$

In the case it is paid just when the option is killed, then its price is given in the next section, remark 9 and also in sect. 6.4.

5.3. Asymmetric Rebates

We here compute separately the value of the upper rebate, which is paid only if the upper barrier is hit first, and of the lower one, paid only if the lower barrier is hit first. As previously, we let τ_h^λ (resp. τ_ℓ^λ) be the first time the process $X_t = \lambda t + W_t$ reaches the level h (resp. ℓ) and $\tau_{h,\ell}^\lambda = \min(\tau_h^\lambda, \tau_\ell^\lambda)$.

Let $\varphi_{h,\ell,+}^\lambda(t)$ (resp. $\varphi_{h,\ell,-}^\lambda(t)$) be the P.D.F. of τ_h^λ (resp. τ_ℓ^λ) conditioned by $\tau_h^\lambda < \tau_\ell^\lambda$ (resp. $\tau_\ell^\lambda < \tau_h^\lambda$), that is :

$$\begin{aligned}
\varphi_{h,\ell,+}^\lambda(t) &= \frac{1}{dt}\,\mathcal{P}\left(\tau_h^\lambda \in [t, t+dt] \text{ and } \tau_h^\lambda < \tau_\ell^\lambda\right)\\
\varphi_{h,\ell,-}^\lambda(t) &= \frac{1}{dt}\,\mathcal{P}\left(\tau_\ell^\lambda \in [t, t+dt] \text{ and } \tau_\ell^\lambda < \tau_h^\lambda\right)
\end{aligned}$$

The sum of these two functions is the P.D.F. of the exit time $\tau_{h,\ell}^\lambda$:

$$\begin{aligned}
\varphi_{h,\ell,+}^\lambda(t) + \varphi_{h,\ell,-}^\lambda(t) &= \frac{1}{dt}\,\mathcal{P}\left(\tau_{h,\ell}^\lambda \in [t, t+dt]\right)\\
&= -\frac{d}{dt}\int_\ell^h \psi_{h,\ell,t}^\lambda(x)\,dx
\end{aligned}$$

where $\psi_{h,\ell,t}^\lambda$ is defined by (4.2). We know that $\psi_{h,\ell,t}^\lambda$ satisfies the Kolmogorov equation :

$$\frac{d}{dt}\psi_{h,\ell,t}^\lambda = \frac{1}{2}\frac{d^2}{dx^2}\psi_{h,\ell,t}^\lambda - \lambda\frac{d}{dx}\psi_{h,\ell,t}^\lambda \tag{5.2}$$

Therefore :

$$\begin{aligned}\varphi^\lambda_{h,\ell,+}(t)+\varphi^\lambda_{h,\ell,-}(t) &= -\frac{1}{2}\left(\frac{d\psi^\lambda_{h,\ell,t}}{dx}(h)-\frac{d\psi^\lambda_{h,\ell,t}}{dx}(\ell)\right)+\lambda\left(\psi^\lambda_{h,\ell,t}(h)-\psi^\lambda_{h,\ell,t}(\ell)\right)\\ &= \frac{1}{2}\left(\frac{d\psi^\lambda_{h,\ell,t}}{dx}(\ell)-\frac{d\psi^\lambda_{h,\ell,t}}{dx}(h)\right)\end{aligned}$$

because $\psi^\lambda_{h,\ell,t}(h)=\psi^\lambda_{h,\ell,t}(\ell)=0$. It is not difficult to see, by perturbing separately h and ℓ, that one can identify each term :

$$\varphi^\lambda_{h,\ell,+}(t)=-\frac{1}{2}\frac{d\psi^\lambda_{h,\ell,t}}{dx}(h) \qquad\qquad \varphi^\lambda_{h,\ell,-}(t)=\frac{1}{2}\frac{d\psi^\lambda_{h,\ell,t}}{dx}(\ell)$$

One has :

$$\begin{aligned}\frac{d\psi^\lambda_{h,\ell,t}}{dx}(\ell) &= e^{-\frac{\lambda^2 t}{2}+\lambda x}\left(\lambda\,\psi_{h,\ell,t}(\ell)+\frac{d}{dx}\psi_{h,\ell,t}(\ell)\right)\\ &= e^{-\frac{\lambda^2 t}{2}+\lambda x}\frac{d}{dx}\psi_{h,\ell,t}(\ell)\\ &= -\frac{2}{t}\,e^{-\frac{\lambda^2 t}{2}+\lambda\ell}\sum_{n=-\infty}^{+\infty}(\ell+2n\delta)\,g_t(\ell+2n\delta)\end{aligned}$$

and

$$\frac{d\psi^\lambda_{h,\ell,t}}{dx}(h)=-\frac{2}{t}\,e^{-\frac{\lambda^2 t}{2}+\lambda h}\sum_{n=-\infty}^{+\infty}(h+2n\delta)\,g_t(h+2n\delta)$$

The price of the Upper Rebate Double Knock-Out option, paid at the end, can now be computed by replacing, for each term, h by $h+2n\delta$ in (3.8) and (3.6) :

$$\mathrm{URDKO}_{\mathbf{end}}(S,L,H,T) = e^{-rT}\int_0^T \varphi^\lambda_{h,\ell,+}(t)\,dt$$

$$=\sum_{n=-\infty}^{+\infty}\left(e^{-rT-2\lambda n\delta}\,N\!\left(-\frac{h+2n\delta}{\sqrt{T}}+\lambda\sqrt{T}\right)+e^{-rT+2\lambda(h+n\delta)}\,N\!\left(-\frac{h+2n\delta}{\sqrt{T}}-\lambda\sqrt{T}\right)\right)$$

If paid at Knock-Out time, one has, setting $\rho=\sqrt{\lambda^2+2r}$:

$$\mathrm{URDKO_{KO}}(S,L,H,T) \;=\; \int_0^T e^{-rt}\,\varphi^{\lambda}_{h,\ell,+}(t)\,dt$$

$$= \sum_{n=-\infty}^{+\infty}\left(e^{-\rho(h+2n\delta)+\lambda h}\,N\!\left(-\frac{h+2n\delta}{\sqrt{T}}+\rho\sqrt{T}\right)+e^{\rho(h+2n\delta)+\lambda h}\,N\!\left(-\frac{h+2n\delta}{\sqrt{T}}-\rho\sqrt{T}\right)\right) \tag{5.3}$$

Symmetrically, the Lower Rebate Double Knock-Out, paid at the end, is given by :

$$\mathrm{LRDKO_{end}}(S,L,H,T) \;=\; e^{-rT}\int_0^T \varphi^{\lambda}_{h,\ell,-}(t)\,dt$$

$$= \sum_{n=-\infty}^{+\infty}\left(e^{-rT-2\lambda n\delta}\,N\!\left(\frac{\ell+2n\delta}{\sqrt{T}}-\lambda\sqrt{T}\right)+e^{-rT+2\lambda(\ell+n\delta)}\,N\!\left(\frac{\ell+2n\delta}{\sqrt{T}}+\lambda\sqrt{T}\right)\right)$$

and, if paid at Knock-Out time, one has, with the same value of ρ :

$$\mathrm{LRDKO_{KO}}(S,L,H,T) \;=\; \int_0^T e^{-rt}\,\varphi^{\lambda}_{h,\ell,-}(t)\,dt$$

$$= \sum_{n=-\infty}^{+\infty}\left(e^{-\rho(\ell+2n\delta)+\lambda\ell}\,N\!\left(\frac{\ell+2n\delta}{\sqrt{T}}-\rho\sqrt{T}\right)+e^{\rho(\ell+2n\delta)+\lambda\ell}\,N\!\left(\frac{\ell+2n\delta}{\sqrt{T}}+\rho\sqrt{T}\right)\right) \tag{5.4}$$

Remark 9. *Adding up (5.3) and (5.4) provides the price of a symmetric rebate which is paid as soon as the option is knocked out.*

6. The BOOST

The BOOST (Banking on Over All STability) is an option that was introduced into the market by Société Générale in 1994. It is characterised by a corridor $[L,H]$. As soon as one of the barriers L or H is reached, the option ends and the buyer receives an amount proportional to the *time* (or to the number of days if discretised) the underlying spent inside the corridor since its issuance. Therefore, after having "lived" for some time, the BOOST has a price made of two parts. The first one is not model dependant and corresponds to the time already spent in the corridor, while the second one is the risk-neutral expectation of the remaining lifetime. For security reasons, a limit to the lifetime has been imposed. If the underlying stays in the corridor until the limit, then the BOOST pays, at expiry, the amount corresponding to this maximum time (as if exit were forced at this date, if it did not happen before). One can also see the BOOST as the time increasing rebate of a double knock-out option.

There are at least three ways to get a closed form formula for the BOOST. One of them is to compute the Green function of the Black-Scholes P.D.E., but the series one gets this way converge slowly. The second is to integrate the value of a Corridor Option with respect to its maturity. This method leads to an integral which is not formally computable by standard means (although we here show that a solution exists). In this section, we give another method to get the BOOST price, through Laplace transforms[9]. This formula is a double infinite series, but which converges extremely fast (like $e^{-\alpha n^2}$: the n-th term measures the probability that a Brownian path alternately hits the upper and the lower barriers n times, similarly to double knock-out options). Practically, three or four terms are enough to get a 10^{-6} precision. The same computation provides the value of the rebate of a double barrier option which is paid as soon as the option knocks out.

6.1. Exit time of a Brownian motion from a corridor

Let us consider the following random process :

$$X_t = \lambda\, t + W_t$$

where W_t is a standard Brownian motion and λ a scalar.

Let h and ℓ denote the barriers and assume $\ell < 0 < h$. Like in previous sections, we set $\delta = h - \ell$ and :

$$\tau_h^\lambda \;=\; \inf\{\, t\,,\; X_t = h\,\}$$

[9]It is not obvious that the series coming from the Green function and that given by the following method converge to the same value. This has been shown by S. Eddé in a personal communication from MUREX. It is left as an exercise to the reader to show that integrating the corridor formula leads to the same value.

$$\tau_\ell^\lambda = \inf\{\, t\,,\ X_t = \ell\,\}$$

$$\tau_{h,\ell}^\lambda = \inf\,(\,\tau_h^\lambda\,,\,\tau_\ell^\lambda\,)$$

We shall first evaluate :

$$T_\lambda(h,\ell) = \mathbf{E}(\tau_{h,\ell}^\lambda)$$

Let

$$p = \mathcal{P}(\,\tau_h^\lambda < \tau_\ell^\lambda\,)$$

Thanks to the fundamental theorem of martingales, one has :

$$\mathbf{E}(X_{\tau_{h,\ell}^\lambda}) = \lambda\, T_\lambda(h,\ell) = h\,p + \ell\,(1-p)$$

On the other hand, for any σ , the process :

$$Y_t^\sigma = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

is a martingale. Taking $\sigma = -2\lambda$, and considering the stopping time $\tau_{h,\ell}^\lambda$, we get :

$$Y_t^{-2\lambda} = e^{-2\lambda X_t}$$

and

$$\mathbf{E}(Y_{\tau_{h,\ell}^\lambda}^{-2\lambda}) = 1 = p\,e^{-2\lambda h} + (1-p)\,e^{-2\lambda\ell}$$

thus

$$p = \frac{e^{-2\lambda\ell} - 1}{e^{-2\lambda\ell} - e^{-2\lambda h}}$$

Consequently

$$T_\lambda(h,\ell) = \frac{\left(e^{-2\lambda\ell} - 1\right)h + \left(1 - e^{-2\lambda h}\right)\ell}{\lambda\left(e^{-2\lambda\ell} - e^{-2\lambda h}\right)}$$

Remark 10. *If $\lambda \to 0$, one finds the simple formula :*

$$T_0(h,\ell) = -h\ell\ .$$

6.2. Distribution of $\tau^{\lambda}_{h,\ell}$ and BOOST without time limit

We shall now compute the Laplace transform of the distribution of $T^{\lambda}_{h,\ell}$. For every $r > 0$, one has :

$$\mathbf{E}(e^{-r\tau^{\lambda}_{h,\ell}}) = p\,A_r + (1-p)\,B_r$$

where

$$A_r = \mathbf{E}(e^{-r\tau^{\lambda}_{h}} \mid \tau^{\lambda}_{h} < \tau^{\lambda}_{\ell}) \qquad B_r = \mathbf{E}(e^{-r\tau^{\lambda}_{\ell}} \mid \tau^{\lambda}_{\ell} < \tau^{\lambda}_{h})$$

But if the number $\sigma \in \mathbf{R}$ satisfies :

$$\sigma\,\lambda + \frac{1}{2}\sigma^2 = r \tag{6.1}$$

then

$$Y^{\sigma}_t = e^{\sigma X_t - rt}$$

and

$$\mathbf{E}(Y^{\sigma}_{\tau^{\lambda}_{h,\ell}}) = 1 = p\,e^{\sigma h} A_r + (1-p)\,e^{\sigma \ell} B_r$$

Let us first consider the case where :

$$r \neq -\frac{1}{2}\lambda^2$$

and denote by σ_1 and σ_2 the two solutions to the equation (6.1) :

$$\begin{aligned} \sigma_1 &= -\lambda + \sqrt{\lambda^2 + 2r} \\ \sigma_2 &= -\lambda - \sqrt{\lambda^2 + 2r} \end{aligned}$$

The conditional expectations A_r and B_r are given by the linear system:

$$\left\{ \begin{aligned} p\,e^{\sigma_1 h} A_r + (1-p)\,e^{\sigma_1 \ell} B_r &= 1 \\ p\,e^{\sigma_2 h} A_r + (1-p)\,e^{\sigma_2 \ell} B_r &= 1 \end{aligned} \right.$$

leading to:

$$\left\{ \begin{aligned} A_r &= \frac{e^{\lambda h}\ \sinh(\sqrt{\lambda^2+2r}\,|\ell|)}{p \sinh(\sqrt{\lambda^2+2r}\,(h-\ell))} \\ B_r &= \frac{e^{\lambda \ell}\ \sinh(\sqrt{\lambda^2+2r}\,h)}{(1-p) \sinh(\sqrt{\lambda^2+2r}\,(h-\ell))} \end{aligned} \right.$$

These expressions provide the Laplace transform of $T^{\lambda}_{h,\ell}$:

$$\mathbf{E}(e^{-r\tau^{\lambda}_{h,\ell}}) \;=\; \frac{e^{\lambda\ell}\sinh(\sqrt{\lambda^2+2r}\,h)-e^{\lambda h}\sinh(\sqrt{\lambda^2+2r}\,\ell)}{\sinh(\sqrt{\lambda^2+2r}\,\delta)} \tag{6.2}$$

By differentiating the Laplace transform with respect to r, we obtain the value of the BOOST without time limit :

$$\mathbf{E}(\,\tau^{\lambda}_{h,\ell}\,e^{-r\tau^{\lambda}_{h,\ell}}) = -\frac{\partial}{\partial r}\mathbf{E}(e^{-r\tau^{\lambda}_{h,\ell}})$$

Letting $\rho = \sqrt{\lambda^2+2r}$ we get the formula :

$$\mathbf{E}(\tau^{\lambda}_{h,\ell}e^{-r\tau^{\lambda}_{h,\ell}}) \;=\;$$

$$\frac{\delta\left(e^{\lambda\ell}\sinh\rho h - e^{\lambda h}\sinh\rho\ell\right)\cosh\rho\delta - \left(h\,e^{\lambda\ell}\cosh\rho h - \ell\,e^{\lambda h}\cosh\rho\ell\right)\sinh\rho\delta}{\rho\sinh^2\rho\delta} \tag{6.3}$$

and, when $\rho\to 0$, that is, if $r=-\frac{1}{2}\lambda^2$, then :

$$\mathbf{E}(\tau^{\lambda}_{h,\ell}e^{-r\tau^{\lambda}_{h,\ell}}) = \frac{\delta}{3}\left(he^{\lambda\ell}-\ell e^{\lambda h}\right) - \frac{h^3e^{\lambda\ell}-\ell^3e^{\lambda h}}{3\delta}$$

Assume now that S_t is a process that satisfies the following diffusion equation :

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW$$

where μ and σ are *constants* and W_t a standard Brownian motion. We set :

$$X_t = \frac{1}{\sigma}\,\log\frac{S_t}{S_0}$$

and we have :

$$dX = \lambda\,dt + dW \qquad\qquad \lambda = \frac{\mu}{\sigma} - \frac{\sigma}{2}$$

The process S_t respectively crosses the barriers L and H, $L < S_0 < H$, if X_t respectively crosses h and ℓ given by :

$$\ell = \frac{1}{\sigma} \log \frac{L}{S_0} \qquad\qquad h = \frac{1}{\sigma} \log \frac{H}{S_0}$$

By inputting these values of λ, h and ℓ in formula (6.3), we get the value of the unlimited BOOST written on S_t with barriers L and H.

Remark 11. *Assume that the process Z_t satisfies the following diffusion equation :*

$$dZ = \mu \, dt + \sigma \, dW$$

with $Z_0 = z$. The value of the unlimited BOOST written on Z_t with barriers h and ℓ, $\ell < x < h$, is obtained by replacing in the previous formula h by $\dfrac{h-z}{\sigma}$, ℓ by $\dfrac{\ell - z}{\sigma}$ and λ by $\dfrac{\mu}{\sigma}$.

Remark 12. *The formula (6.2) provides the price of the (symmetric) rebate of a double knock-out option with infinite maturity, that would be paid when knocking out. See sect. 6.4 for the rebate of double barrier options with finite maturity.*

6.3. BOOST with a time limit

Let $M > 0$ and :

$$\tau_M = \inf \left(\tau^{\lambda}_{h,\ell} \, , M \right)$$

be the stopping time of a BOOST with time limit M. Our aim is to compute the value of such a BOOST, that is :

$$\mathbf{E}(e^{-r\tau_M} \, \tau_M) = \int_0^M t \, e^{-rt} P^{\lambda}_{h,\ell}(t) \, dt + M \, e^{-rM} \int_M^{\infty} P^{\lambda}_{h,\ell}(t) \, dt$$

where $\varphi^{\lambda}_{h,\ell}$ is the distribution density of $\tau^{\lambda}_{h,\ell}$.

We shall apply the following Laplace transform inversion formula to compute the function $P^{\lambda}_{h,\ell}$:

$$\tilde{f}(x) = \int_0^{\infty} e^{-xt} f(t) \, dt = \frac{\sinh(\alpha\sqrt{x})}{\sinh(\beta\sqrt{x})}$$

when

$$f(t) = \frac{2\pi}{\beta^2} \sum_{n=1}^{\infty} (-1)^n \, n \, e^{-\frac{n^2\pi^2 t}{\beta^2}} \sin \frac{n\pi\alpha}{\beta}$$

This formula is a limit case of Heaviside expansion theorem on Laplace transforms (see [1, p.1021, 29.2.20]) : let $a_1, \dots, a_n \in \mathbb{C}$ and P be a polynomial of degree less than n , then the inverse Laplace transform of the rational function $P(x)/Q(x)$, where $Q(x) = \prod_{k=1}^{n}(x - a_k)$, is :

$$\sum_{k=1}^{n} \frac{P(a_k)}{Q'(a_k)} \; \exp(a_k t)$$

applied to the product expansion (see [1, p.85, 4.5.68]) :

$$\sinh z = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right)$$

From the identity :

$$\widetilde{(e^{-ct} f)}(s) = \tilde{f}(s + c)$$

and from equation (6.2), we get :

$$\varphi^{\lambda}_{h,\ell}(t) = \frac{\pi}{\delta^2} \, e^{-\frac{\lambda^2}{2} t} \sum_{n=1}^{\infty} (-1)^{n-1} \, n \, e^{-\frac{n^2\pi^2 t}{2\delta^2}} \left(e^{\lambda \ell} \sin \frac{n\pi h}{\delta} - e^{\lambda h} \sin \frac{n\pi \ell}{\delta} \right)$$

In order to compute $\mathbf{E}(e^{-r\tau_M} \, \tau_M)$, we observe that the previous series for $\varphi^{\lambda}_{h,\ell}(t)$ converges very fast (like e^{-an^2}) provided t is not too small. As we already know the value of the unlimited BOOST, it is sufficient to compute its difference with the limited one. Let I_M be defined by :

$$\mathbf{E}(\tau_M e^{-r\tau_M}) = \mathbf{E}(\tau^{\lambda}_{h,\ell} e^{-r\tau^{\lambda}_{h,\ell}}) - I_M$$

One has :

$$I_M \;\; = \;\; \int_M^{\infty} \left(t \, e^{-rt} - M \, e^{-rM} \right) \varphi^{\lambda}_{h,\ell}(t) \, dt$$

and if we set $u_n = \dfrac{n^2\pi^2}{2\delta^2} + \dfrac{\lambda^2}{2}$ then :

$$\int_M^\infty t\, e^{-rt} \varphi_{h,\ell}^\lambda(t)\, dt \quad =$$

$$\frac{\pi}{\delta^2} \sum_{n=1}^\infty (-1)^n\, n\, \frac{1+(u_n+r)M}{(u_n+r)^2}\, e^{-(u_n+r)M} \left(e^{\lambda\ell} \sin\frac{n\pi h}{\delta} - e^{\lambda h} \sin\frac{n\pi\ell}{\delta} \right)$$

$$M\, e^{-rM} \int_M^\infty \varphi_{h,\ell}^\lambda(t)\, dt =$$

$$\frac{\pi}{\delta^2} \sum_{n=1}^\infty (-1)^n\, n\, \frac{M}{u_n}\, e^{-(u_n+r)M} \left(e^{\lambda\ell} \sin\frac{n\pi h}{\delta} - e^{\lambda h} \sin\frac{n\pi\ell}{\delta} \right)$$

We conclude that :

$$I_M = \frac{\pi}{\delta^2} \sum_{n=1}^\infty (-1)^n\, n\, \frac{rM(1+\frac{r}{u_n})-1}{(u_n+r)^2}\, e^{-(u_n+r)M} \left(e^{\lambda\ell} \sin\frac{n\pi h}{\delta} - e^{\lambda h} \sin\frac{n\pi\ell}{\delta} \right)$$

Remark 13. *This expression for I_M is not singular when $\frac{\lambda^2}{2} + r = 0$.*

Remark 14. *By modifying the drift λ, one can compute the value of a BOOST with barriers that exponentially depend on time : $H(t) = H_0 e^{\theta t}$, $L(t) = L_0 e^{\theta t}$.*

Remark 15. *When $M \to \infty$, the first term of the series provides the following equivalent for I_M :*

$$I_M \sim \frac{\pi^2}{\delta^2} \frac{1 - rM(1+\frac{r}{u_1})}{(u_1+r)^2}\, e^{-(u_1+r)M} \left(e^{\lambda\ell} \sin\frac{\pi h}{\delta} - e^{\lambda h} \sin\frac{\pi\ell}{\delta} \right)$$

with

$$u_1 = \frac{\pi^2}{2\delta^2} + \frac{\lambda^2}{2}$$

this equivalent is already a very good approximation when $M > \delta^2$.

Remark 16. *If $M > \dfrac{u_1}{r(u_1+r)}$, and in particular, if $M > \dfrac{1}{r}$, then $I_M < 0$ which means that the "unlimited" BOOST is cheaper than the limited one. This phenomenon comes from the fact that the BOOST reaches such a value that postponing the payment date by one day costs more than the day value.*

6.4. Symmetric Rebate of Double Barrier Options

The price of rebates of double barrier options which are paid as soon as the option is knocked out is given by the expectation of the discount factor $e^{-r\tau_M}$:

$$\mathrm{RDKO_{KO}}(S,L,H) = \mathbf{E}(e^{-r\tau_M}) = \int_0^M e^{-rt}\varphi_{h,\ell}^{\lambda}(t)\,dt$$

Where M denotes the option maturity. As for the BOOST, we shall compute its value by its complement to the "infinite" case given by formula (6.2). One has :

$$\mathbf{E}(e^{-r\tau_M}) = \mathbf{E}(e^{-r\tau_{h,\ell}^{\lambda}}) - J_M$$

with

$$\begin{aligned} J_M &= \int_M^{\infty} e^{-rt}\,\varphi_{h,\ell}^{\lambda}(t)\,dt \\ &= \frac{\pi}{\delta^2}\sum_{n=1}^{\infty}(-1)^n\,n\,\frac{e^{-(u_n+r)M}}{u_n+r}\left(e^{\lambda\ell}\sin\frac{n\pi h}{\delta} - e^{\lambda h}\sin\frac{n\pi\ell}{\delta}\right) \end{aligned}$$

where, as previously :

$$u_n = \frac{n^2\pi^2}{2\delta^2} + \frac{\lambda^2}{2}$$

It is left to the reader that this formula is consistent with the results of sect. 5.3 (see footnote 9).

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