HYBRID CALIBRATION OF INTEREST RATE MODELS: IMPLIED and STATISTICAL

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Problem No 1: Model Choice

Characteristics of the model

- Number of factors
- Markov property ⇔ "recombining" tree
  i.e. model with state variables
- Volatility structure:
  - with respect to time
  - with respect to rates

Examples of Models

Model	Nb. Factors	Markov	Depend. / time	Depend. / rates
Vasicek	1	✓	×	Normal
Black-Karasinsky	1	✓	✓	Log-normal
Hull-White	1	✓	✓	Normal
Heath-Jarrow-Morton	*n*	×	✓	Free
Cox-Ingersoll-Ross	1	✓	×	Quadratic
Brace-Musiela	*n*	×	✓	Log-normal
Duffie-Kan, El Karoui	*n*	✓	✓	Quadratic
" String "	∞	✓	✓	Free

Problem No 2: Calibration

Find model parameters that reproduce market prices of liquid assets.

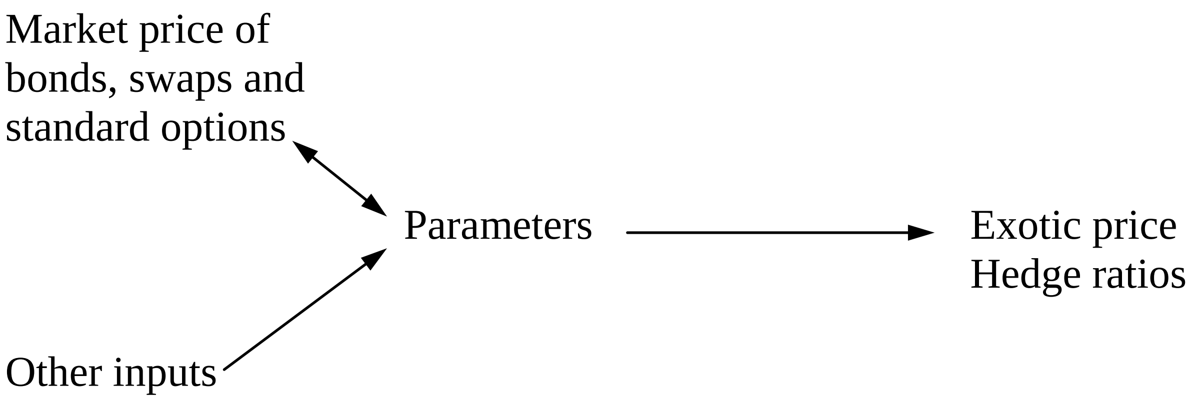
- Bond prices, cash rates, swap rates ⇒ Discount factors $DF(t,T)$
- Option prices:
  - Caplets
  - Swaptions

Model Parameters: Current practice

- Drift: According to model
  - ∗ Imposed by arbitrage
  - ∗ Calibrated upon discount factors at the origin of times
- Volatility: Function of time and rates
  - ∗ Calibrated on caplet prices
  - ∗ Calibrated on swaption prices
- Correlations: Inputs or statistical

Aim of Calibration

- Price and hedge exotic options
- Optimise portfolio management (hedging, trading)



Two types of sensitivity:
- With respect to model variables
- With respect to model parameters

"Model" price = Financial result (profit/loss) — within the model — of the dynamic hedging of risk on variables.

Dilemma of Parameters that Model Assumes Fixed

If they vary, one hopes that:

1. *Price* = $\mathbf{E}_{\text{actual}}$[*Financial result of dynamic hedging*]
2. Variance reduction

For (1), the following is needed:

- A correct estimation of the <u>expectation</u> of what will be the actual value of parameters in the future,
- The option price must depend linearly — or almost — on parameters.

For (2), the following is needed:

- Either that parameters vary little and slowly,
- Or that the option price barely depends on them,
- Or hedge the option against parameter variations.

In the latter case, one is sensitive to the volatility of parameters and to the option convexity *with respect to parameters* (generalised Black-Scholes-Kolmogorov equation).

The belief that a model can be entirely calibrated on market data and a dynamic hedging strategy blindly applied, without decision-making, is <u>pure illusion.</u>

At best, missing parameters are provided by statistics. If the option is <u>convex/concave</u> w.r.t. a volatile parameter, this parameter <u>should be made stochastic</u> in the model.

Model Complexity

- **Low-dimensional model:**
  - Easy to calibrate
  - Limited range of options:
    - American, Bermudan
    - Regular barriers
    - No reverse barriers

- **High-dimensional model:**
  - Needs statistical data
  - Heavy computations (calibration <u>and</u> pricing)
  - Approximate closed forms
  - Reliable prices and hedges

Market Data for Calibration

The model describes the term structure diffusion over a period $[t_0, t_{max}]$. Market data are asset prices <u>at the initial date</u> t_0 .

- **Yield Curve**

  It provides interest rate *drift* through the arbitrage equation :

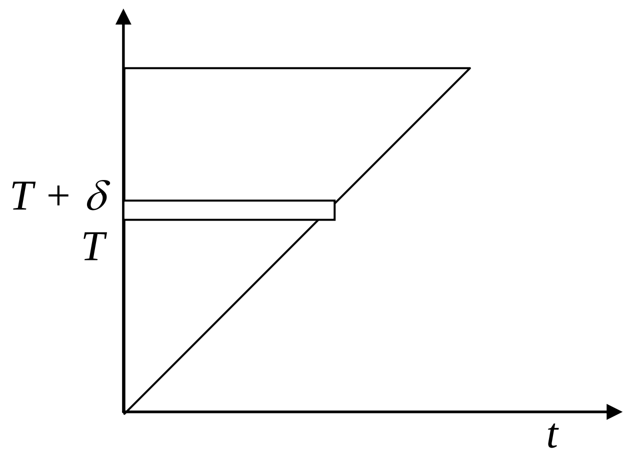
$$B(t_0, T') = \mathrm{E}\left[\exp\left(-\int_{t_0}^{T} r(s)\,ds\right) B(T, T')\right]$$

- **Caplets**

  Let $\sigma(t,T,\delta)$ be the instantaneous volatility at time t of the F.R.A. rate on the period $[T,T+\delta]$. This function of the two variables (t,T) is called “Volatility triangle”.

  The price of a caplet on a δ-maturity rate is given by a Black-Scholes like formula involving the integral:

$$\int_{t_0}^{T} \sigma(t,T,\delta)^2\, dt$$



Swaptions

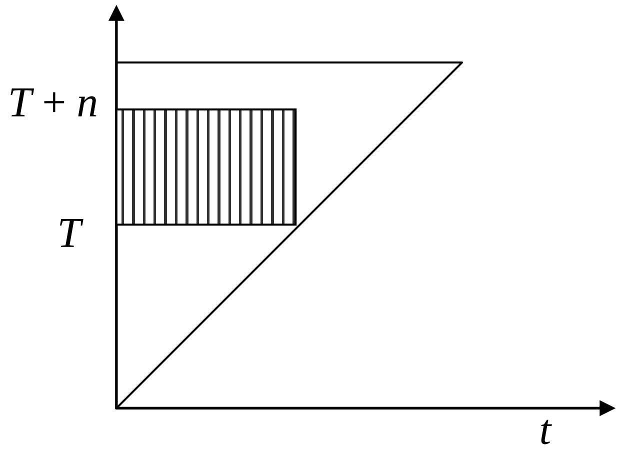
Because of the usually high correlation between rates of various maturity, the instantaneous volatility at date t of the forward swap rate starting at T with n years maturity, is approximately:

$$\sigma_{\text{swap}}(t,T,n) \approx \sum_{i=0}^{n/\delta} \lambda_i \, \sigma(t, T + i\delta, \delta)$$

where coefficients λ_i are the sensitivities of the swap rate with respect to " buckets " $[T + i\delta,\ T + (i+1)\delta]$. In practice, the swap rate volatility $\sigma_{\text{swap}}(t,T,n)$ is slightly lower than this combination because "buckets" are not fully correlated. The price of a T-maturity European swaption on an n years maturity swap is again approximately given by Black-Scholes-Kolmogorov formula involving the integral:

$$\int_{t_0}^{T} \sigma_{\text{swap}}(t,T,n)^2 \, dt \approx \int_{t=t_0}^{T} \left(\int_{u=T}^{T+n} \sigma(t,u)\, du \right)^2 dt$$

- Caplet prices provide the square average volatility on a *line* of the triangle $\sigma(t,T)$, $t \in [t_0,T]$
- Swaption prices provide a (t-square, T-linear)-average volatility over a *rectangle* $\sigma(t,t')$, $t \in [t_0,T]$, $t' \in [T, T+n]$
- Correlations affect these averages.



Numerical Methods for Calibration

Most common methodology in Interest Rate market

1. Compute statistical correlations,
2. Find factors relative shape to match statistical correlations,
3. Find volatility triangle to match market price of caplets and swaptions,
4. Loop through steps 2 and 3 until convergence.

Select:

- a series of caplets $C_1,\ldots,C_m$ with maturity sampled over the given period $[t_0, T_{\max}]$,
- a series of swaptions $S_1,\ldots,S_p$ identified by (*option maturity*, *rate maturity*).

Minimise a Least Square Criterion:

$$\Phi(\alpha_1,\ldots,\alpha_k)=\sum_{i=1}^{m}\mu_i\left(C_i(\alpha_1,\ldots,\alpha_k)-\hat{C}_i\right)^2+\sum_{i=1}^{p}\nu_i\left(S_i(\alpha_1,\ldots,\alpha_k)-\hat{S}_i\right)^2$$

- α_i are model parameters that are to be calibrated (others are assumed to be known).
- Weights μ_i et ν_i must be chosen to provide a uniform accuracy on volatilities.
  Φ is not necessarily convex.
- Non-linear optimisation algorithms (among others):
  - Levenberg-Marquardt
  - Broyden-Fletcher-Goldfarb-Shanno
- Stabilise by penalisation: $\Phi_\varepsilon(\alpha_1,\ldots,\alpha_k)=\Phi_0(\alpha_1,\ldots,\alpha_k)+\varepsilon\,Q(\alpha_1,\ldots,\alpha_k)$

Model Term Structure and Stationarity

- If correlations are fully statistically estimated then:

Vanilla Options Term Structure $\Rightarrow$ Model Parameters Term Structure

- The Market can only predict a time-dependent probabilistic behaviour for the near future (typically a few months)

$\Rightarrow$ Model must be stationary for $t \geq t_1 = t_0 + \Delta t$ ($\Delta t \approx 3$ months)

⇨ *Solution 1*

- Rigid model term structure:
  - One or two time periods, constant parameters on each period

  or

  - Fast converging term structure (e.g. decreasing exponential)

⇨ *Solution 2*

- Approximate matching of statistical correlations
- Penalisation of parameter variation through time for $t \geq t_1$

- Global Calibration vs. Bootstrap

Parameters α_i are *constant numbers*, not *functions*.
In order to calibrate a *time dependent parameter*, use:

- Parametric representation (e.g. polynomial, etc.)
- Piecewise constant, linear
- Splines

Large number of parameters $\Rightarrow$ Slow optimisation

Boostrap

1. Divide the maturity range in *slices* $[T_0,T_1],\ldots,[T_{n-1},T_n]$,
2. Piecewise constant, linear, spline representation,
3. Calibrate options maturing in $[T_0,T_1]$ with parameters in this interval,
4. For $k = 2,\ldots,n$, assuming parameters are known in $[T_0,T_{k-1}]$, use parameters in $[T_{k-1},T_k]$ in order to calibrate options maturing in this interval.
    - Advantage: Speed
    - Drawback: Oscillations

Oscillation reduction

- H^1-penalty: Penalise parameter changes from one time slice to the next.
  Not slower than regular bootstrap but may introduce a bias.
- H^2-penalty: Penalise convexity in parameter variation over 3 consecutive slices.
  Almost as accurate as global parameterisation, but much faster.

1 Factor Markov Models

- Advantage:
  - Simplicity
  - Computation speed
  - Almost closed form formulas

- Drawback:
  - Parameter instability
  - Rigidity of volatility triangle

This model is adapted to instruments/portfolios that are localised in maturity, or as a sub-model of a more complex one, in which interest rate risk involves one maturity only. Examples:

- ❑ Long-term options (FX, index, etc.)
- ❑ Options on a sequence of futures
- ❑ Convertible, callable bonds, etc.

Such a model is entirely determined by the short rate diffusion process:

$$dr = \alpha(t,r)\, dt + \sigma(t,r)\, dW$$

Then by the arbitrage equation:

$$B(t,T) = \mathbf{E}_{\text{Risk-neutral}}\left[\exp\left(-\int_t^T r(s)\, ds\right)\right]$$

Examples

- Affine e.g. Hull-White:

$$dr = (\beta(t) - \alpha(t)\, r)\, dt + \sigma(t)\, dW$$

The short rate $r(t)$ is an *Ornstein-Uhlenbeck* process. It is Gaussian and one can compute in closed forms expectations, variances and covariances.

- *Caplets* are given by Black-Scholes formula.
- *Swaptions* are given in closed form after solving a 1-dimensional equation for the critical short rate r^* above/below which the option is exercised.

- Quadratic e.g. Cox-Ingersoll-Ross:

$$dr = \left(\gamma(t) + \beta(t)\sqrt{r} - \alpha(t)\, r\right)dt + \sigma(t)\sqrt{r}\, dW$$

- Log-normal e.g. Black-Derman-Toy:

$$d \log r = \left(\beta(t) - \alpha(t) \log r\right)dt + \sigma(t)\, dW$$

This model diverges in continuous time.

Limitations of 1-factor Markov models: One matches market prices of caplets and of a 1-dim series of swaptions, but not of a custom 2-dim set.

Non Markov 1-Factor Models

Examples: 1-factor Gaussian Heath-Jarrow-Morton
1-factor Brace-Gatarek-Musiela

- It is possible to match a whole triangle of swaption market prices, but calibration is unstable and dynamic hedging is not more efficient than with Markov models.
- Markov dimension is only bounded by the total number of rates diffused

$\Rightarrow$ Potentially high complexity

Affine model, 1-Factor H.J.M., Markov in Dimension 2 :

$$\begin{cases} dr = r'\,dt + \sigma(t)\,dW \\ dr' = (\beta(t) - \alpha(t)r')\,dt + \sigma'(t)\,dW \end{cases}$$

- r is the short rate
- r' is the forward spot rates curve slope at origin
- (r, r') is a 2-dimensional Ornstein-Uhlenbeck
- Caplets in closed forms, swaptions approximated by closed forms (moment matching)

Affine models can have a customary number of factors, as well as Markov dimensions (at least the number of factors).

Multi-Factor Models

- **Affine** e.g. Heath-Jarrow-Morton: Rates are Gaussian and one can compute in closed forms expectations, variances and covariances.
    - *Caplets* are given by Black-Scholes formula
    - *Swaptions* are given in approximate closed form by moment matching

- **Quadratic** e.g. Duffie-Kan, El Karoui-Durand: Rates are given as a quadratic polynomial of Gaussian variables, volatilities are affine functions of these variables.
    - *Caplets* are given in closed form
    - *Swaptions* are given in approximate closed form

- **Log-normal** e.g. Brace-Gatarek-Musiela, Jamshidian: A Libor rate with given positive tenor (e.g. 3 months) follows a log-normal process. This model applies in continuous time. All options in approximate closed form.
    - All options are given in approximate closed form

Numerical Algorithms

***Markov dimension* 2 - 4**

- ❑ Finite differences, finite elements: Crank-Nicholson, A.D.I., Galerkin
    - ➢ American and Bermudan options
    - ➢ Convertible bonds
    - ➢ Callable bonds, prepayment option (MBS)

Markov dimension* > 4 *(whatever the number of factors)

- ❑ Monte-Carlo for *exotic options*
  Although approximate closed forms exist for caplets and European swaptions.
    - ➢ Barrier and Rainbow options
    - ➢ Ratchet, Digitals
    - ➢ Whole portfolios of standard options
- ❑ Rising literature on the pricing of options with early exercise feature in a Monte-Carlo framework:
    - Barraquand-Martineau
    - Brodie-Glasserman
    - Longstaff-Schwartz
    - Dupire
    - …

Adaptive Calibration

Based on daily use and historical convergence

Instead of splitting parameters between:

- Statistically estimated
- Market implied

We calibrate every day the whole set of parameters such that:

- Market data are matched exactly,
- Parameter shift from yesterday is minimal in some well-chosen Euclidean norm.

Intuitive idea: Among all possibilities of market data fitting, we chose the less stochastic set of parameters

$\Rightarrow$ Impact of parameter volatility on option pricing and hedging is minimal

Kahlman Filtering

Assume:

- $\theta_1,\ldots,\theta_q$ are unobservable stochastic processes, with unknown evolution !
- $P(t, \theta_1,\ldots,\theta_q)$ is the price of the exotic option
- $X_i(t, \theta_1,\ldots,\theta_q)$, $i = 1,\ldots,n < q$ are hedging instruments

The *hedging vector* $(X_1,\ldots,X_n) \in \Delta \subset \mathbf{R}^n$ and on the boundary $\partial\Delta$, one has:

$$P = \Psi(X_1,\ldots,X_n)$$

Where Ψ is a known function (asymptotic value).

***Question* 1 :**

Find $P(t, X_1,\ldots,X_n$, other observables) and hedging strategies so as to optimise some criterion
e.g. expectation of utility $\Rightarrow$ Hamilton-Jacobi-Bellman

***Question* 2 :**

Fix time t and find $P(X_1,\ldots,X_n$, other observables) in order to minimise both variance and squared drift.
Because the portofolio composition is also an unknown for a market maker.