

## ON MEASURING HEDGE FUND RISK

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**Abstract.** We propose a methodology for estimating hedge fund risk, which consists of two steps: first, regressing the return on non-linear functions of each single factor and second, merging together the obtained estimates taking into account the dependence between different factors. This enables us to use as a lever the fact that we have more information on factors than on hedge funds. As a by-product, we obtain a new approach to non-linear hedge fund replication.

In order to assess the risk of a hedge fund, one could simply estimate some risk measure (for example, standard deviation or Value at Risk) using its return history. A deeper approach, however, would be to relate the hedge fund return to risk driving factors like equity, bond, currency, commodity, and volatility indices, credit spreads, etc. This provides the advantages of:

- taking into account predictions about the future moves of the factors;
- using the long time series available for factors and thus providing more accurate risk predictions;
- estimating the joint risk of different hedge funds, i.e. estimating the risk of portfolios of hedge funds;
- the possibility of extrapolating back the hedge fund history, using the data for the factors.

The most classical approach to relating hedge fund returns to risk factors is the linear regression on a collection of factors. This is closely connected with hedge fund replication (see Fung and Hsieh, 1997a, Hasanhodzic and Lo, 2007). However, the findings of Hasanhodzic and Lo (2007) show that such a method explains only 15–20% of variance (i.e. risk) in individual hedge fund returns. One of the reasons is the well-recognized non-linearity in the dependence of hedge fund returns on the factors (see Agarwal and Naik, 2004, Fung and Hsieh, 1997b, 2001, Lo, 2001).

In order to capture this non-linearity, one might take options on factors as additional factors, as is done in Agarwal and Naik (2004). However, the following problem then arises. The hedge fund returns typically have quite a short observation history; for example, if a fund has a 2-year history, then we will have two dozens of observations of its return. On the other hand, the number of non-linear factors might well exceed the number of observations, and then it is impossible to do the regression of the return on all the factors due to overfitting. One way to overcome this problem is to extract from the factors a few relevant ones and thus avoid overfitting. For this, one might use some “blind” method such as “stepwise regression” or “matching pursuit”. For example, Agarwal and Naik (2004) choose as the factors one index and four options on this index.

In this paper, we propose another way to regress the return on the collection of non-linear factors. Let  $R$  denote the hedge fund return over a fixed time period and  $X_1, \dots, X_N$  be the increments of factors over the same time period. The methodology we propose consists of two steps:

1. Regress the return on the non-linear functions of each *single* factor, i.e. for each  $n = 1, \dots, N$ , find the best approximation of  $R$  by a sum of the form

$$\varphi_n(X_n) = a_n + b_n X_n + \sum_{i=1}^{I_n} c_{ni} (X_n - K_i)^+, \quad (1)$$

where  $K_1, \dots, K_{I_n}$  are some chosen strikes of traded call options on  $X_n$ .

2. Join together the functions  $\varphi_1(X_1), \dots, \varphi_N(X_N)$ , taking into account the dependence between the factors, to produce a non-linear approximation of  $R$  by a function of all factors.

The methodology for Step 1 is clear: we perform simply a linear regression of  $R$  on a factor and a family of options. The methodology for Step 2 is not obvious since different factors are dependent. Performing Step 2 is the main objective of this paper.

To conclude the introduction, let us mention two advantages of the proposed method over the method of choosing few relevant factors:

- When performing Step 2, we are taking as an input the joint distribution of  $X_1, \dots, X_N$  estimated from the long time high-frequency time series available. In contrast, when regressing the return on the set of relevant factors, one would take only the monthly increments of factors, and only for the period of the fund history. Thus, we use as a lever the fact that we have much more information on factor history than that of the fund.
- We use the known structure of the factor set, namely the fact that cells are made of non-linear versions of the same factor. This allows to avoid accidental factors that might arise when using “blind” factor selection methods.

Let us also mention two nice features of the approach we are proposing, of which more will be said later:

- The solution we obtain has the form of a sum of functions of single factors, which provides a new approach to non-linear hedge fund replication.
- The solution is linear in  $R$ , so that if one has performed Steps 1–2 for each single hedge fund, he/she gets free the solution for any portfolio of these funds. This makes the approach convenient for measuring the risk of hedge fund portfolios and hence, for constructing optimal ones, which is the fund of funds problem.

## Mathematical Formulation

Let us formalize the problem of Step 2. We are given an  $N$ -dimensional probability density  $p(x_1, \dots, x_N)$ , which is the joint density of  $X_1, \dots, X_N$ . We are also given non-linear functions  $\varphi_1(X_1), \dots, \varphi_N(X_N)$  meaning the best non-linear approximations of the hedge fund return by functions of each single factor.<sup>1</sup> The problem is to recover from this information the best non-linear approximation of  $R$  by a non-linear function  $\varphi(X_1, \dots, X_N)$  of all the factors.

First, let us establish the relationship between  $\varphi$  and each  $\varphi_n$ . If  $\varphi(X_1, \dots, X_N)$  is the result of the non-linear regression of  $R$  on all the factors, then  $\varphi_n(X_n)$  should be the result of the non-linear regression of  $\varphi(X_1, \dots, X_N)$  on the  $n$ -th factor. This means that the function  $\varphi_n$  should deliver the minimum in the optimization problem

$$\int \dots \int (\varphi(x_1, \dots, x_N) - \psi(x_n))^2 dx_1 \dots dx_N \longrightarrow \min,$$

where the minimization is over all the non-linear functions  $\psi(x_n)$ . Throughout the paper, all the limits of integration are from  $-\infty$  to  $+\infty$ , and we will skip them. The solution to

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<sup>1</sup>Functions  $\varphi_n$  are closely connected with *factor risks* introduced independently in literature by Cherny and Madan (2005) and in the industry by Risk Data.

the above problem has a well-known form, so we get the equation<sup>2</sup>

$$\varphi_n(x_n) = \frac{\int \dots \int \varphi(x_1, \dots, x_N) p(x_1, \dots, x_N) dx_1 \dots dx_{n-1} dx_{n+1} \dots dx_N}{\int \dots \int p(x_1, \dots, x_N) dx_1 \dots dx_{n-1} dx_{n+1} \dots dx_N}. \quad (2)$$

In other words, the information we have about the unknown function  $\varphi$  is its integrals over the hyperplanes orthogonal to the coordinate axes.

This information is however insufficient to recover the function  $\varphi$ . As an example, let

$$p(x_1, x_2) = \frac{1}{2\pi} \exp\left\{-\frac{x_1^2 + x_2^2}{2}\right\}$$

be the standard Gaussian density in the two-dimensional space and consider  $\varphi_1(x) = \varphi_2(x) = 0$ . Then both functions  $\varphi(x_1, x_2) = 0$  and  $\varphi(x_1, x_2) = x_1 x_2$  satisfy (2). It is clear that the non-uniqueness effect will be present for any natural  $p$ .

In order to get a unique solution, one should impose additional restrictions on the unknown function  $\varphi$ . We are proposing to look for a function, which is “the most moderate one” as measured by its second moment. Then the problem becomes

$$\begin{cases} \int \dots \int \varphi^2(x_1, \dots, x_N) p(x_1, \dots, x_N) dx_1 \dots dx_N \longrightarrow \min, \\ \varphi \text{ satisfies (2) for any } n. \end{cases} \quad (3)$$

## Gaussian Distribution

In this section, we present an explicit solution of (3) for the case when  $p$  is the density of a non-degenerate Gaussian distribution. We will assume that, for any  $n$ ,

$$\int \dots \int x_n p(x_1, \dots, x_N) dx_1 \dots dx_N = 0, \quad (4)$$

$$\int \dots \int x_n^2 p(x_1, \dots, x_N) dx_1 \dots dx_N = 1. \quad (5)$$

This is sufficient for applications since the main application we have in mind is the Gaussian copula, and it can always be transformed into such a density  $p$ . We will denote by  $C$  the covariance matrix of  $p$ , i.e.

$$C_{nk} = \int \dots \int x_n x_k p(x_1, \dots, x_N) dx_1 \dots dx_N.$$

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<sup>2</sup>Another way to explain this equation is as follows. The best non-linear estimate of  $R$  by  $X_n$  is given by the conditional expectation

$$\varphi_n(x) = \mathbb{E}[R | X_n = x];$$

the best non-linear estimate of  $R$  by all  $X_1, \dots, X_N$  is the conditional expectation

$$\varphi(x_1, \dots, x_N) = \mathbb{E}[R | X_1 = x_1, \dots, X_N = x_N].$$

Using now the tower property of conditional expectations, we arrive at (2).

If functions  $\varphi_n$  satisfy (2), then

$$\begin{aligned} & \int \dots \int \varphi_n(x_n) p(x_1, \dots, x_N) dx_1 \dots dx_N \\ &= \int \dots \int \varphi(x_1, \dots, x_N) p(x_1, \dots, x_N) dx_1 \dots dx_N. \end{aligned}$$

Hence, a necessary condition for the existence of a solution of (3) (which is now assumed to hold) is that there exists a constant  $E$  such that, for any  $n$ ,

$$\int \dots \int \varphi_n(x_n) p(x_1, \dots, x_N) dx_1 \dots dx_N = E. \quad (6)$$

Recall that the *Hermite polynomials* are defined recurrently as:  $H_0(x) = 1$ ,

$$H_n(x) = \frac{1}{\sqrt{n}} \left( x H_{n-1}(x) - \frac{d}{dx} H_{n-1}(x) \right), \quad n = 1, 2, \dots$$

As an example,

$$\begin{aligned} H_1(x) &= x, \\ H_2(x) &= (x^2 - 1)/\sqrt{2}, \\ H_3(x) &= (x^3 - 3x)/\sqrt{6}, \\ H_4(x) &= (x^4 - 6x^2 + 3)/\sqrt{24}. \end{aligned}$$

Consider the values  $(a_{nm}, n = 1, \dots, N, m = 1, 2, \dots)$  defined as

$$a_{nm} = \frac{1}{\sqrt{2\pi}} \int \varphi_n(x) H_m(x) e^{-x^2/2} dx.$$

Denote by  $C^m$  the  $m$ -th componentwise power of  $C$ , i.e.  $C_{nk}^m = (C_{nk})^m$ . It is known that this matrix is non-degenerate.<sup>3</sup> So, we can find the values  $(\alpha_{nm}, n = 1, \dots, N, m = 1, 2, \dots)$  defined through the following linear systems:

$$\begin{cases} C_{11}^m \alpha_{1m} + \dots + C_{1N}^m \alpha_{Nm} = a_{1m}, \\ \dots \\ C_{N1}^m \alpha_{1m} + \dots + C_{NN}^m \alpha_{Nm} = a_{Nm}. \end{cases}$$

**Theorem 1.** *The solution of (3) is unique and is given by*

$$\varphi(x_1, \dots, x_N) = E + \sum_{n=1}^N \sum_{m=1}^{\infty} \alpha_{nm} H_m(x_n).$$

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<sup>3</sup>A result going back to Jacobi states that if  $(A_{nk})_{n,k=1}^N$  and  $(B_{nk})_{n,k=1}^N$  are symmetric positively definite non-degenerate matrices, then their componentwise product  $(A_{nk} B_{nk})_{n,k=1}^N$  has the same properties. For a simple probabilistic proof, we refer to Cherny, Douady, and Molchanov (2008).

The proof is given in the Appendix.

Let us now consider a particular case when each  $\varphi_n$  is linear. In view of (4) and (6), this means that  $\varphi_n(x) = E + a_n x$  with some constants  $E, a_1, \dots, a_N$ . It is well known that Hermite polynomials have the property

$$\frac{1}{\sqrt{2\pi}} \int x^M H_m(x) e^{-x^2/2} dx = 0 \quad \text{for } m > M.$$

Therefore, we get from Theorem 1

**Corollary 1.** *If  $\varphi_n(x) = E + a_n x$ , then the solution of (3) is given by*

$$\varphi(x_1, \dots, x_N) = E + \sum_{n=1}^N \alpha_n x_n,$$

where  $\alpha_1, \dots, \alpha_N$  are found through the system

$$\begin{cases} C_{11}\alpha_1 + \dots + C_{1N}\alpha_N = a_1, \\ \dots \\ C_{N1}\alpha_1 + \dots + C_{NN}\alpha_N = a_N. \end{cases}$$

More generally, we get

**Corollary 2.** *If each  $\varphi_n$  is a polynomial of degree  $M$ , then the solution of (3) is given by*

$$\varphi(x_1, \dots, x_N) = E + \sum_{n=1}^N \sum_{m=1}^M \alpha_{nm} H_m(x_n).$$

Corollary 1 shows that in the Gaussian case our technique is consistent with the linear regression technique. Namely, consider random variables  $R, X_1, \dots, X_N$ , where  $X_1, \dots, X_N$  have the joint density  $p$  satisfying the conditions of this section. Then the linear regression of  $R$  on  $X_n$  is given by the function  $\varphi_n(x) = E + a_n x$ , where  $E = E[R]$  and

$$a_n = \frac{E[RX_n]}{E[X_n^2]} = E[RX_n].$$

The linear regression of  $R$  on all  $X_1, \dots, X_N$  is given by

$$\varphi(x_1, \dots, x_N) = E + \sum_{n=1}^N \alpha_n x_n,$$

where  $\alpha_n$  are exactly the same as in Corollary 1. In other words, if  $X_1, \dots, X_N$  are jointly Gaussian, then the linear regression estimate of  $R$  by  $X_1, \dots, X_N$  can be recovered from the one-dimensional linear regressions, and our technique does this in particular.

## Gaussian Copula

Let us now consider the case when the density  $p$  corresponds to a Gaussian copula, i.e. there exist strictly increasing functions  $f_n$  such that the vector

$$(\tilde{X}_1, \dots, \tilde{X}_N) = (f_1^{-1}(X_1), \dots, f_N^{-1}(X_N)) \quad (7)$$

has a Gaussian density  $\tilde{p}$ . Conditions (4), (5) imposed on  $\tilde{p}$  mean that  $E\tilde{X}_n = 0$  and  $E\tilde{X}_n^2 = 1$ , from which the functions  $f_n$  are determined uniquely. Denote

$$\tilde{\varphi}_n(x) = \varphi_n(f_n(x)).$$

It is clear that a function  $\varphi(x_1, \dots, x_N)$  solves (3) if and only if the function

$$\tilde{\varphi}(x_1, \dots, x_N) = \varphi(f_1(x_1), \dots, f_N(x_N))$$

is the solution of (3) corresponding to  $\tilde{p}$  and  $\tilde{\varphi}_1, \dots, \tilde{\varphi}_N$ . The latter solution has been given above.

So, the proposed methodology of hedge fund risk estimation in the case when the joint law of the factor increments  $X_1, \dots, X_N$  is described by a Gaussian copula, consists of the following steps:

1. Do linear regression (1) of  $R$  on each factor and some options on the factor to obtain a non-linear estimate  $\varphi_n(X_n)$ .
2. Estimate from the time series of factors the joint density  $p(x_1, \dots, x_N)$  of  $X_1, \dots, X_N$  and find the functions  $f_n$  such that the random vector (7) has a joint Gaussian density with  $E[\tilde{X}_n] = 0$ ,  $E[\tilde{X}_n^2] = 1$ .
3. Fix a number  $M \in \mathbb{N}$  (for example, 30) and find the coefficients  $(a_{nm}; n = 1, \dots, N, m = 1, \dots, M)$  given by:

$$a_{nm} = \frac{1}{\sqrt{2\pi}} \int \varphi_n(f_n(x)) H_m(x) e^{-x^2/2} dx.$$

4. Find the values  $(\alpha_{nm}; n = 1, \dots, N, m = 1, \dots, M)$  by solving the linear systems

$$\begin{cases} C_{11}^m \alpha_{1m} + \dots + C_{1N}^m \alpha_{Nm} = a_{1m}, \\ \dots \\ C_{N1}^m \alpha_{1m} + \dots + C_{NN}^m \alpha_{Nm} = a_{Nm}, \end{cases}$$

where  $C_{nk}^m = (\text{cov}(\tilde{X}_n, \tilde{X}_k))^m$ .

5. Then the solution of (3) exists, is unique, and is given by

$$\varphi(x_1, \dots, x_N) = E + \sum_{n=1}^N \sum_{m=1}^{\infty} \alpha_{nm} H_m(f_n(x_n)),$$

where  $E$  is defined by (6). Then  $\varphi(X_1, \dots, X_N)$  is the proposed non-linear approximation of  $R$  by a function of all factors.

The above function  $\varphi$  is a sum of one-dimensional functions of single factors. So, our approximation of  $R$  might be realized through a portfolio of single factors and options on them. This is a very convenient feature as it provides a new approach to the non-linear hedge fund replication.

Another nice feature of the proposed solution is the linearity in  $R$ . Let  $R^1, \dots, R^M$  denote the returns of different hedge funds. Let  $\varphi_n^m(X_n)$  be the regression (1) for  $R^m$ . Denote by  $\varphi^m(x_1, \dots, x_N)$  the above given solution of (3) corresponding to  $\varphi_1^m, \dots, \varphi_n^m$ .

Consider now the return of a portfolio of the above hedge funds:

$$R = \sum_{m=1}^M \mu^m R^m,$$

where  $\mu^1, \dots, \mu^M$  are positive numbers with  $\sum_m \mu^m = 1$ . Then the regression (1) for  $R$  is

$$\varphi^m(X_n) = \sum_{m=1}^M \mu^m \varphi_n^m(X_n).$$

Clearly, the above solution of (3) corresponding to  $\varphi_1, \dots, \varphi_N$  has the form

$$\varphi(x_1, \dots, x_N) = \sum_{m=1}^N \mu^m \varphi^m(x_1, \dots, x_N).$$

In other words, the solution of (3) corresponding to a weighted average of portfolios is the weighted average of solutions corresponding to individual portfolios.<sup>4</sup>

## Three Types of Risk

Let  $R, X_1, \dots, X_N$  be the same as above. Here we impose no assumptions on the density of  $X_1, \dots, X_N$ . Let  $\varphi_n(X_n)$  be the best non-linear estimate of  $R$  by functions of  $X_n$ , which is the conditional mean

$$\varphi_n(x) = E[R | X_n = x].$$

Although  $p$  is arbitrary, the solution of (3) will typically exist, be unique, and have the form

$$\varphi(x_1, \dots, x_N) = \sum_{n=1}^N h_n(x_n)$$

with some functions  $h_1, \dots, h_N$ ; see Cherny, Douady, and Molchanov (2008). As an example, in the case of a Gaussian copula, the functions  $h_n$  have explicitly been provided in the previous section.

The function  $\varphi(X_1, \dots, X_N)$  obtained as the solution of (3) is not the true best non-linear estimate of  $R$  by a function of all the factors. Indeed, we had to recover it from the best estimates by each single factor, and the complete recovery is impossible, as

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<sup>4</sup>Both the linearity in  $R$  and the fact that the solution of (3) is a sum of functions of single factors are valid not only for a Gaussian copula but for any model; see Cherny, Douady, and Molchanov, (2008).



discussed above. Therefore, the function  $\varphi(X_1, \dots, X_N)$  will be different from the true best non-linear estimate  $\psi(X_1, \dots, X_N)$ , which is given by the conditional mean

$$\psi(x_1, \dots, x_N) = \mathbb{E}[R | X_1 = x_1, \dots, X_N = x_N].$$

The latter function exists theoretically, but, as discussed in the introduction, it cannot be estimated practically due to the lack of data. In order to understand better the relationship between the above two functions, let us consider the decomposition

$$R = R^1 + R^2 + R^3,$$

where

$$\begin{aligned} R^1 &= \varphi(X_1, \dots, X_N), \\ R^2 &= \psi(X_1, \dots, X_N) - \varphi(X_1, \dots, X_N), \\ R^3 &= R - \psi(X_1, \dots, X_N). \end{aligned}$$

**Proposition 1.** *The random variables  $R^1, R^2, R^3$  are uncorrelated.*

The proof is given in the Appendix.

The function  $\varphi$  is a sum of monomials, i.e. functions of single variables  $x_1, \dots, x_N$ . Therefore, we might say that the random variable  $R^1$  is responsible for the *monomial risk*. The random variable  $R^2$  captures the non-linearity in  $\psi$  coming from the non-monomial terms such as  $x_n x_k$ . So, we might say that  $R^2$  is responsible for the *cross-term risk*. Finally,  $R^3$  captures the uncertainty in  $R$  not explained by  $X_1, \dots, X_N$ , i.e.  $R^3$  is responsible for the *idiosyncratic risk*. Thus, the above decomposition of  $R$  might be interpreted as:

$$\text{Risk} = \text{Monomial risk} + \text{Cross-term risk} + \text{Idiosyncratic risk}.$$

In these terms, the topic of this paper is recovering the monomial risk.

## Summary

We have proposed a new method for assessing hedge fund risk. It consists of: first, linearly regressing the hedge fund return on each single factor and some options on it; second, joining together the obtained non-linear estimates. An explicit way of doing that has been provided for the case when the factors follow a Gaussian copula. The proposed methodology led us to decomposing hedge fund risk into three components: monomial risk, cross-term risk, and idiosyncratic risk. As a by-product, we have obtained a new way of non-linear hedge fund replication. The method proposed is linear in the hedge fund return, which makes it convenient for assessing the risk of portfolios of hedge funds.

## Appendix: The Proofs

**Proof of Theorem 1.** The well-known property of Hermite polynomials states that

$$\mathbb{E}[H_m(X_n)H_l(X_k)] = \begin{cases} 0 & \text{if } m \neq l, \\ C_{nk}^m & \text{if } m = l. \end{cases}$$

Hence, for any  $n$ , we have

$$\mathbb{E}[\varphi(X_1, \dots, X_N)H_m(X_n)] = \sum_{k=1}^N C_{nk}^m \alpha_{km} = a_{nm} = \mathbb{E}[\varphi_n(X_n)H_m(X_n)], \quad m = 1, 2, \dots$$

The random variables  $H_1(X_n), H_2(X_n), \dots$  form an orthonormal basis in the space  $L_n$ , which consists of random variables of the form  $f(X_n)$  with  $\mathbb{E}[f(X_n)] = 0$ ,  $\mathbb{E}[f^2(X_n)] = 1$ . So, we see that

$$\text{Pr}_{L_n} \varphi(X_1, \dots, X_N) = \varphi_n(X_n),$$

where “Pr” denotes the orthogonal projection. According to the known properties of conditional expectations, this means that

$$\varphi_n(x) = \mathbb{E}[\varphi(X_1, \dots, X_N) | X_n = x],$$

which is equivalent to (2).

In order to prove the  $L^2$ -minimality of  $\varphi$ , take another function  $\tilde{\varphi}$  satisfying (2) for any  $n$ . Then

$$\text{Pr}_{L_n} \tilde{\varphi}(X_1, \dots, X_N) = \varphi_n(X_n) = \text{Pr}_{L_n} \varphi(X_1, \dots, X_N).$$

This means that the difference  $\tilde{\varphi}(X_1, \dots, X_N) - \varphi(X_1, \dots, X_N)$  is orthogonal to the sum of spaces  $L_1 + \dots + L_N$ . On the other hand,  $\varphi(X_1, \dots, X_N)$  belongs to this sum. As a result,

$$\mathbb{E}[\varphi^2(X_1, \dots, X_N)] \leq \mathbb{E}[\tilde{\varphi}^2(X_1, \dots, X_N)],$$

which is exactly the minimality of  $\varphi$  we need.

**Proof of Proposition 1.** Without loss of generality,  $E = 0$ . A function  $\tilde{\varphi}$  satisfies (2) if and only if

$$\text{Pr}_{L_n} \tilde{\varphi}(X_1, \dots, X_N) = \varphi_n(X_n),$$

where  $L_n$  are the same as above. The definition of  $\varphi_n$  means that

$$\varphi_n(X_n) = \text{Pr}_{L_n} R.$$

Thus, a function  $\tilde{\varphi}$  satisfies (2) if and only if for any  $n$ ,

$$\text{Pr}_{L_n} \tilde{\varphi}(X_1, \dots, X_N) = \text{Pr}_{L_n} R,$$

which is, in turn, equivalent to:

$$\text{Pr}_L \tilde{\varphi}(X_1, \dots, X_N) = \text{Pr}_L R,$$

where  $L$  is the closure of the sum  $L_1 + \dots + L_N$ . Now, it is clear that the solution of (3) is:

$$\varphi(X_1, \dots, X_N) = \text{Pr}_L R.$$

From the well-known property of conditional expectations, we can write

$$\psi(X_1, \dots, X_N) = \Pr_{L'} R,$$

where  $L'$  denotes the space of square integrable random variables of the form  $f(X_1, \dots, X_N)$  with  $E[f(X_1, \dots, X_N)] = 0$  and  $E[f^2(X_1, \dots, X_N)] < \infty$ . Clearly,  $L$  is included in  $L'$ . Thus, we get

$$R^1 = \Pr_L R, \quad R^2 = \Pr_{L'} R - \Pr_L R, \quad R^3 = R - \Pr_{L'} R,$$

from which the result is clear.

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